Chapter 24: The Expected Utility Model

24.1: Introduction

In this chapter we introduce an empirically-relevant model of preferences for representing behaviour under conditions of risk – the Expected Utility Model. Like the Discounted Utility Model (which we used to describe intertemporal preferences) this model is not only a good empirical approximation to reality, but it also has compelling normative properties. These, as with the Discounted Utility Model, refer to the consistency of the behaviour of an individual with such preferences, and the potential inconsistency of the behaviour of individuals with preferences that do not satisfy the model. We can easily illustrate the point with a discussion of a key axiom of the theory – known as the Independence Axiom. Let $C$ and $D$ be risky choices and suppose we have an individual who says that he or she prefers $C$ to $D$. Let $E$ be any other risky choice. The Independence Axiom then states that he or she should also prefer the gamble (call it $G$) which yields $C$ with probability $p$ and $E$ with probability $(1-p)$ to another gamble (call it $H$) which yields $D$ with probability $p$ and $E$ with probability $(1-p)$. Why? Because under $G$ the individual either gets $C$ or $E$, and with $H$ the individual either gets $D$ or $E$. Now it has been assumed that the individual prefers $C$ to $D$ - so he or she should prefer $G$ to $H$. Depending on which he or she chooses he or she either gets $C$ or $D$ (and he or she prefers the first of these) or gets $E$ (whichever he or she has chosen). Does this seem 'reasonable'? Why is it a criterion of consistency? Take, for simplicity $p=0.5$, and suppose that the $G$ and $H$ gambles are played out by tossing a fair coin: if it lands Heads, he or she gets $C$ or $D$, depending on which of $G$ and $H$ he or she has chosen; and if it lands Tails, he or she gets $E$. Now consider an individual who does not obey the Independence Axiom and says that while he or she prefers $C$ to $D$ he or she prefers $H$ to $G$. He starts out preferring $H$ to $G$, and hence choosing $H$. Suppose we now toss the coin and it lands Heads. He or she now has $D$ – but he or she would have had $C$ - which he or she prefers to $C$- if he or she had chosen $G$. At this point the individual would want to change their decision. The violation of the Independence Axiom implies a sort of built-in inconsistency in the preferences.

24.2: The Expected Utility Model

This is used to describe preferences under risk. Let us recall the decision situation and the notation from chapter 23. We are considering a situation of risk in which the decision maker does not know \textit{ex ante} which state of the world will occur. However he or she can list the various possibilities and can attach probabilities to them. For simplicity here we simply assume two possible states of the world, state 1 and state 2, with respective probabilities $\pi_1$ and $\pi_2$. \textit{Ex ante}, when choosing between various decisions, the decision maker does not know which of these states will occur. \textit{Ex post}, one and only one of the states will occur.

We denote by $c_1$ the individual’s consumption/income (we use these terms interchangeably) if state of the world 1 occurs and by $c_2$ individual’s consumption/income if state of the world 2 occurs. The individual must choose \textit{ex ante} between various uncertain bundles ($c_1$, $c_2$). \textit{Ex post}, the individual gets one of $c_1$ or $c_2$ depending upon which state of the world has occurred. We want to describe preferences over \textit{ex ante} risky consumption bundles ($c_1$, $c_2$). The Expected Utility Model is specified as follows\textsuperscript{1}.

\textsuperscript{1} It can be derived formally from a set of axioms which crucially include the Independence Axiom. A sketch of the proof – which you are not expected to know – is provided in the Mathematical Appendix to this chapter.
\[ U(c_1, c_2) = \pi_1 u(c_1) + \pi_2 u(c_2) \quad (24.1) \]

The probabilities are given by the problem so the only element that needs to be specified is the function \( u(\cdot) \). This is sometimes known as the Neumann-Morgenstern utility function, after a mathematician (von Neumann) and an economist (Morgenstern) who refined the theory. Given the function \( u(\cdot) \), which tells how much utility is got from some amount of consumption, the explanation of (24.1) is clear: with probability \( \pi_1 \) state 1 happens and the individual consumes \( c_1 \) from which he or she gets utility \( u(c_1) \); with probability \( \pi_2 \) state 2 happens and the individual consumes \( c_2 \) from which he or she gets utility \( u(c_2) \). So if we consider the utility that an individual expects to get from the \textit{ex ante} risky bundle \((c_1, c_2)\), it is clear that this is given by the right-hand side of equation (24.1). It seems reasonable to suppose that an individual chooses between risky bundles on the basis of the utility that they expect to get from them: if they expect to get more utility from one bundle than another they should choose this first bundle rather than the other.

24.3: Indifference Curves with the Expected Utility Model

We have now specified the Expected Utility Model. In the next section we explore its implications. But first, as we are going to use the framework for analysis developed in chapter 23, we need to look at the properties of the indifference curves in \((c_1, c_2)\) space implied by this model. To drive this result we need a little mathematics which is provided in the Mathematical Appendix to this chapter.

The slope of an indifference curve is
\[ \frac{\text{dc}_1}{\text{dc}_2} = -\frac{\pi_1}{\pi_2} \frac{\text{du}(c_1)/\text{dc}_1}{\text{du}(c_2)/\text{dc}_2} \quad (24.2) \]
where \( \text{du}(c)/\text{dc} \) indicates the slope of the function \( u(c) \). The slope is negative so the indifference curves are downward sloping. Moreover if \( u \) is \textit{concave} then, as we move down and rightwards along an indifference curve, \( c_1 \) is rising while \( c_2 \) is falling, and hence \( \text{du}(c_1)/\text{dc}_1 \) is falling while \( \text{du}(c_2)/\text{dc}_2 \) is rising, and so the magnitude of the slope \( \text{dc}_2/\text{dc}_1 \) is falling. From this it follows that if \( u \) is concave then the indifference curves are convex. If however \( u \) is \textit{linear} then both \( \text{du}(c_1)/\text{dc}_1 \) and \( \text{du}(c_2)/\text{dc}_2 \) are constant and so the slope of the indifference curves are constant – that is they are linear. If we continue this line of argument with a convex function \( u \), then we get the following result:

\textit{If } u \text{ is concave, linear or convex then the indifference curves in } (c_1, c_2) \text{ space are convex, linear or concave.}

One final result is of particular importance. If, in (24.2) we put \( c_1 = c_2 \) we get that the slope of an indifference curve is \( -\pi_1/\pi_2 \). Calling the line \( c_1 = c_2 \) the ‘certainty line’ we get the important result that:

\textit{Along the certainty line the slope of every indifference curve of an individual with Expected Utility Model preferences is equal to } -\pi_1/\pi_2.

If we connect the result above – that the utility function \( u \) is concave, linear or convex according as the indifference curves are convex, linear or concave – with our interpretation in chapter 23 of convex, linear and concave indifference curves as representing those of a risk-averse, risk-neutral and risk-loving individual – it follows immediately that an individual with a concave, linear or convex utility function must be risk-averse, risk-neutral or risk-loving. Let us check this out in the next section.
24.4: Risk Aversion and Risk Premia

Consider an individual with a concave utility function \( u \) as in figure (24.1). For those of you who want to verify the detail algebraically you might like to know that the utility function used here is given by \( u(c) = \frac{(1 - e^{-0.03c})}{(1 - e^{-3.3})} \). This is an example of what we term in section 24.5 a constant absolute risk averse utility function. You do not need to do algebra, however, as all the information necessary is contained in the figures.

Suppose this individual is offered a 50-50 gamble between 30 and 70 – that is, with probability 0.5 the individual will get 30 and with probability 0.5 the individual will get 70. How does this individual evaluate this risky bundle? We could say that the expected income from the risky bundle is 50, but we know that, if the individual has Expected Utility preferences, then the evaluation is on the basis of the expected utility rather than the expected income. Now we know the utility function so we can calculate the expected utility of the risky prospect (30, 50). Consumption 30 gives utility approximately equal to 0.616 while consuming 70 gives utility approximately equal to 0.912. So this risky prospect gives utility 0.616 with probability 0.5 and utility 0.912 with probability 0.5. Hence the expected utility from this prospect is \( 0.5 \times 0.616 + 0.5 \times 0.912 = 0.764 \). The three horizontal lines in figure 24.4 give these utilities: the bottom line is the utility of 30, the top line the utility of 70 and the middle line the expected utility of the risky prospect. (It is in the middle in this example because the two possibilities are equally likely.)

Now we define the certainty equivalent of the risky prospect. It is defined as the amount of money which, if received with certainty, the individual regards as equivalent to the risky prospect. Using our model of Expected Utility the individual regards a sum of money as equivalent to a risky prospect if it gives the same expected utility as the prospect. Now obviously the expected utility of a certain amount is simply equal to the utility of that amount. So the certainty equivalent of the 50-50 risky prospect which gives 30 or 70 (each with equal probability) is given by the following expression, where \( ce \) denotes the certainty equivalent.

\[
u(ce) = 0.5 \times u(30) + 0.5 \times u(70) = 0.764\]

From the figure we see that \( ce \) is approximately equal to 44.5 – because the utility of 44.5 is 0.764.

You will notice that the certainty equivalent of the risky prospect is less than the expected income from the risky prospect. This individual regards the risky prospect as being equivalent to having
44.5 with certainty, and regards having 50 with certainty as being preferable to having the risky prospect. This individual is clearly risk averse.

We now define another concept – that of the risk premium that the individual is willing to pay. This is defined as the difference between the certainty equivalent of the risky prospect and the expected income from the risky prospect. If we look at the figure we see that this risk premium is the distance indicated with an arrow. It is the difference between 50 (the expected income from the prospect) and 44.5 (the certainty equivalent of the prospect) – which is equal to 5.5. The vertical lines in figure 24.1 show these various monetary amounts: the left and right hand vertical lines are the two possible outcomes of the prospect; the line in the middle is the expected outcome of (or income from) the prospect (it is in the middle in this example because the two possible outcomes are equally likely); and the line to the left of it is the certainty equivalent of the prospect. The difference indicated with an arrow is the risk premium.

What is the interpretation of this risk premium? If we took the risk away from the risky prospect we would have a certain return of 50 (the expected income from the prospect). So the risk premium is the maximum amount that the individual would pay to have the risk taken away from the prospect. It tells us how much at most the individual would pay to have certainty rather than risk. As you might be able to anticipate the risk premium depends upon the risky prospect itself and on the shape of the utility function. We will later show that it depends in particular on the concavity of the utility function – the more concave the greater the risk premium, and hence the more risk averse the individual. We shall study this in more detail shortly, but in the meantime let us generalise our definition of the certainty equivalent and the risk premium.

Let us consider a general risky prospect \((c_1, c_2)\) where the probability of \(c_1\) is \(\pi_1\) and the probability of \(c_2\) is \(\pi_2\). The certainty equivalent of this prospect, denoted by \(ce\), is, as before, the amount of money received with certainty that the individual regards as equivalent to the risky prospect. It is given by

\[
u(ce) = \pi_1 u(c_1) + \pi_2 u(c_2)\]  

(24.3)

Be sure that you are clear about this. It means that if the individual is offered a choice between \(ce\) for sure and the risky prospect he or she would say that he or she was indifferent – he or she does not mind which he or she has nor whether someone else chooses for him or her. Furthermore if offered a choice between some amount greater than \(ce\) for sure and the risky prospect, he or she would take the certain amount; and if offered a choice between some amount less than \(ce\) for sure and the risky prospect, he or she would take the risky prospect.

The risk premium, denoted by \(rp\), is defined in general as follows

\[
 rp = (\pi_1 c_1 + \pi_2 c_2) - ce
\]  

(24.4)

It is the difference between the expected income from the prospect and the certainty equivalent of the prospect. The risk premium is the maximum amount of money that the individual would pay to get rid of the riskiness of the prospect and have it replaced by a certainty equal to its expected value.

In the next section we will examine some particular utility functions but in the meantime we should note that the utility function which represents a particular set of preferences under risk is not unique. Indeed it can be shown that if any function represents a set of preferences under risk then so does any other function which is an arbitrary increasing linear transformation of that function. The reason for this is that if the function \(v\) is an increasing linear function of the function \(u\), then the
expected value of \( v \) is the same linear function of the expected value of \( u \). Specifically if \( u \) represents the preferences of the individual then so does \( v \) where \( v \) is defined by \( v = a + bu \) where \( a \) and \( b \) are arbitrary numbers (though \( b \) must be positive). If the expected value of \( u \) represents preferences then so does the expected value of \( v \): if the expected value of \( u \) is higher with one gamble then so must the expected value of \( v \). This means that the scale of the utility function is arbitrary\(^2\).

**24.5: Constant Absolute Risk Aversion**

One popular (and empirically not too bad) utility function is that known as the *constant absolute risk aversion* function. It is given\(^3\) by

\[
\text{u}(c) \propto -\exp(-rc) \tag{24.5}
\]

There is one parameter here – the parameter \( r \) – which is referred to as the *index of absolute risk aversion*. If \( r \) is positive the function (24.5) is concave and the individual is risk-averse; the greater is \( r \) the greater the degree of concavity and the more risk averse is the individual. In the example above we used such a function, with \( r = 0.03 \).

Why is it called the ‘constant absolute risk aversion’ function? Because the risk premium that the individual is willing to pay is independent of the *level* of the prospect. For example, taking the example we have already discussed and portrayed in figure 24.1, this function is of the constant absolute risk aversion form with the parameter \( r \) equal to 0.03. We have already calculated that an individual with this function, faced with the 50-50 prospect \((30, 70)\) would pay a risk premium of 5.5. Now consider the same individual faced with the 50-50 prospect \((5, 45)\). How much would he or she pay as risk premium. Figure (24.2) shows the answer: 5.5! The same is true for the 50-50 prospect \((55, 95)\). So this individual’s risk premium for the 50-50 prospect \((5, 45)\) is 5.5; for the 50-50 prospect \((30, 70)\) is 5.5; for the 50-50 prospect \((55, 95)\) is 5.5. And what is the difference between these 3 prospects? The expected income from the prospect: 25 or 50 or 75. But note very carefully that the *riskiness* of the prospects does not change – the two outcomes are always –20 or +20 with respect to the expected income. So the riskiness of these 3 prospects is the same – that is why the risk premium for someone with constant absolute risk aversion remains unchanged.

\(^2\) If this worries you, think of *temperature*. Its scale is arbitrary (if you are told that the temperature is 80 that means nothing until you know what scale is being used) but we use temperatures all the time.

\(^3\) The words ‘proportional to’ simply reflect the fact that the scale is arbitrary.
If, however, we change the riskiness of the prospect then the risk premium increases.

The risk premium is also bigger for an individual with a more concave utility function – that is one with a higher value of the parameter $r$. This can be seen very clearly in figure 24.4 which shows the risk premium that an individual with $r = 0.5$ would be willing to pay for the 50-50 prospect $(30, 70)$. Compare this figure with figure 24.2 above.

![Figure 24.4: constant absolute risk averse (05)](image1)

### 24.6: Risk Neutrality

This is an important special case. In this the utility function $u$ is linear. The risk premium is zero as the certainty equivalent of any risky prospect is equal to the expected income from the prospect. Figure 24.6 illustrates.

![Figure 24.6: risk neutral](image2)

### 24.7: Constant Absolute Risk Loving

Here we have the function

$$u(c) \text{ proportional to } \exp(r c)$$  \hspace{1cm} (24.6)

There is again one parameter here – the parameter $r$ – which is referred to as the index of absolute risk loving. If $r$ is positive the function (24.5) is convex and the individual is risk-loving; the greater is $r$ the greater the degree of convexity and the more risk loving is the individual.
We can again define the certainty equivalent of any given risky prospect. For a risk-loving individual the certainty equivalent is greater than the expected income from the prospect. Figure 24.8 illustrates – for the case when \( r = 0.03 \). Here the certainty equivalent of the 50-50 prospect \((30, 70)\) is 53.5. We can, once again, define the risk premium as the difference between this and the expected income from the prospect – though in the case of a risk-lover it is the minimum amount that he or she would pay to keep the risky prospect rather than have it replaced by the certainty of its expected value.

If the function becomes more convex the individual becomes more risk-loving and the risk premium rises – the individual is willing to pay more for the risk.

### 24.8: Constant Relative Risk Aversion and Loving

Some empirical evidence suggests that for some individuals their risk premium depends upon the level of their expected income – so that the risk premium declines as the expected income rises. For such individuals the constant absolute risk aversion utility function is not appropriate and a better one may be the constant relative risk aversion utility function. This is specified as follows:

\[
\text{u(c) proportional to } c^{1-r} \quad (24.7)
\]

where, once again, the parameter \( r \) indicates the level of risk-aversion or loving. If \( r \) takes the value 0 then \( u \) is linear and we have the risk-neutral case. If \( r \) is between 0 and 1 then the exponent of \( c \) in the function (24.7) is also between 0 and 1 so that the function is concave and we have a risk-averse individual. Furthermore the closer is \( r \) to 0 the more concave is the function and the more risk-averse is the individual. If \( r \) is negative then the exponent of \( c \) in (24.7) is greater than 1 so the function is convex and we have a risk-loving individual. Furthermore the more negative is \( r \) the more risk-loving the individual.

This utility function has a nice property. Let us denote by \((x, y)\) a risky prospect which pays \( x \) with probability \( \frac{1}{2} \) and pays \( y \) with probability \( \frac{1}{2} \). Then with this utility function the risk premium for \((5, 45)\) is less than the risk premium for \((30, 70)\) which is less than the risk premium for \((55, 95)\) – or more generally the risk premium for \((a-b, a+b)\) falls as \( a \) increases with \( b \) remaining constant.

You may be wondering why it is called the constant relative risk aversion function. This is because the risk premium for \((s(a-b), s(a+b))\) is proportional to the scale \( s \). For example the risk premium for \((15, 35)\) is twice that for \((30, 70)\) and is three times that for \((45, 105)\).
24.9: Optimal Behaviour for an Individual with Expected Utility Preferences

We are now ready to work out what various individuals will do. Consider an example in which the two states of the world are equally likely and where the individual starts at the point \((30, 50)\) – that is without insurance the individual will get consumption/income 30 if state 1 occurs and will get consumption/income 50 if state 2 occurs. Let us suppose that the individual has Expected Utility preferences with a constant absolute risk aversion utility function with \(r = .03\). His or her indifference curves in \((c_1, c_2)\) space are given in figure 24.12. Note that the slope of each and every indifference curve along the certainty line is \(-1\). This is a consequence of the general result that the slope along the certainty line is equal to minus the ratio of the two probabilities – which are, in this case, equal.

We have also inserted in the figure the budget constraint for a fair insurance market. The prices of the two state contingent incomes are both \(\frac{1}{2}\) and so the budget constraint also has slope \(-1\). It immediately follows that the optimal point is on the certainty line at the point \((40, 40)\) – the individual buys 10 units of state 1 contingent income and sells 10 units of state 2 contingent income. Whichever state occurs the individual ends of with 40 units of consumption/income. He or she chooses to be completely insured.

Note that this must be true for any individual with expected utility preferences – as we know that the slope of each and every indifference curve along the certainty line is equal to \(-\pi_1/\pi_2\) and we know that the fair insurance line has the same slope. For example, with \(\pi_1 = 0.4\) and \(\pi_2 = 0.6\) we have figure 24.13.

What about a risk-neutral individual? Well we know that his or her indifference curves are parallel straight lines all with slope \(-\pi_1/\pi_2\) - which is equal to the slope of the fair budget line. So we have figure 24.14 for the case of equal probabilities. The budget constraint coincides with an
indifference curve and the individual is therefore indifferent between all the points along it. Offered
fair insurance – which changes the riskiness but not the expected income of his
consumption/income – the risk-neutral individual is indifferent – because he or she is indifferent to
the risk.

What about a risk-loving individual? He or she has concave indifference curves so his or her
position is as in figure 24.15 for the equal probabilities case.

The optimal point (the highest attainable indifference curve consistent with the budget constraint) is
either at \((80, 0)\) or at \((0, 80)\). The individual either gambles on state 1 happening or on state 2
happening. He or she uses the insurance market to do exactly the opposite of what you might think
that an insurance market is meant to do\(^4\) – to create a riskier prospect which he or she prefers to the
safer prospect.

24.10: Summary

This chapter has been a bit difficult but it contains some important ideas, not least of which is the
definition of Expected Utility preferences.

The Expected Utility model postulates that an ex ante risky bundle is evaluated on the basis of the
expected utility of the various outcomes.

A key component of these preferences is the (Neumann-Morgenstern) utility function of the
individual – defined over consumption/income. This enables us to identify the risk attitude of the
individual.

\(^4\) In fact, many insurance companies ban this kind of deal – the reason being that the individual now has a strong
incentive to try and change the probabilities of the two states.
A risk-averse, -neutral, -loving individual has a concave, linear, convex utility function.

From this we worked out the form of the individual’s indifference curves in \((c_1, c_2)\) space.

A risk-averse, -neutral, -loving individual has convex, linear, concave indifference curves.

We introduced the important concepts of the certainty equivalence of a risky prospect and the risk premium.

The certainty equivalent of a risky prospect is the amount of money received with certainty that the individual regards as equivalent to the risky prospect. The risk premium is the maximum amount of money that the individual would pay to exchange the risky prospect for a certainty with the same expected income.

As far as optimal insurance behaviour is concerned we found some interesting results.

Risk-averse individuals always choose to be fully insured in a fair market, while risk-neutral agents are indifferent and risk-loving agents use the market to take a risky gamble.

We also introduce two important special cases of (Neumann-Morgenstern) utility functions.

The absolute risk premium paid by individuals with a constant absolute risk averse utility functions are independent of the level of the risky prospect (adding a constant to all the outcomes does not change the risk premium).

The relative risk premium paid by individuals with a constant relative risk averse utility functions are independent of the scale of the risky prospect (multiplying by a constant all the outcomes multiplies the risk premium by the same constant).

24.11: How risk-averse are you?

Every year, either in Bari or in York or in both, depending on how I am feeling, I auction off to my students the right to play once with me the following game. “You toss a fair coin once; if it comes down Heads, I will give you £100; if it comes down Tails; I will not give you anything.” I usually do a straightforward English-type auction. I start off at a price of £0 and ask all those who would be willing to pay that price (to play the game with me) to raise their hands. I then progressively raise the price and tell the students to put their hands down when the price gets to the maximum that they are willing to pay to play the game. I tell them that the last person who has his or her hand up will play the game with me, and that that person has to pay the price at which the next-to-the-last person dropped out. I start at £0; usually all the students start with their hands up; by the time that the price has reached £10 quite a few have dropped out; more and more drop out as the price gets higher and higher; by the time it has reached £50 there are usually only a few left; sometimes the penultimate person drops out at a price of more than £50. The last person in pays this price and then he or she tosses the coin. If it lands Heads I give him or her £100; if it lands Tails, I do not give him or her anything; in either case I pocket the price that the individual has paid\(^5\).

\(^5\) If you are interested, I can report that I have always had to pay out £100! Fate is obviously against me.
Why do I do this? To give the students some feel for how risk-averse they are, and to make them realise that people differ in their aversion to risk. Some are very risk-averse – they drop out of the auction at a very low price; some are less risk-averse – they stay in somewhat longer. Some are risk-neutral – they stay in until the price gets to £50. Some are risk-loving – they will stay in above a price of £50. Amongst my students, the vast majority are risk-averse, some very much so, and very few indeed are risk-loving.

At York, I follow up this auction with a tutorial exercise designed to shed more light on how risk-averse people are. The purpose of the exercise is to get students to discover their utility function for choice under risk – the utility function in the Expected Utility theory that we have been studying in this chapter. You can do the same – it is an amusing and instructive exercise. It is also a little difficult – not in a technical sense – but in that it asks you to introspect about yourself. This is something that is remarkably difficult to do.

Suppose your preferences concerning risky choice are in accordance with Expected Utility theory. Then there exists some underlying utility function which can be used to represent your choices in risky situations, in that you will always take that decision which maximises the expected utility you get from the decision. The purpose of this exercise is to find your function. You should note that these functions are personal: your function represents how you behave; if people are different then these functions are different.

We are going to find your function over some range of money. The method can be used over other arguments, but restricting it to money keeps the exposition simple. Let the range of money be from £0 to £100. Assuming that you prefer more money to less, it follows that having £100 is better than having any amount between £0 and £100 and that, in turn, is better than having £0. In the terms of the Mathematical Appendix, £100 is the most preferred and £0 the least preferred. We will assign utilities of 1 and 0 to these respectively. So we start with

\[ u(£100) = 1 \quad \text{and} \quad u(£0) = 0 \]

These are two points on your utility function. You could start to draw a graph with money from £0 to £100 along the horizontal axis, and with utility from 0 to 1 on the vertical axis. Insert the two points given by the equations above.

To find other points on your utility function, we use a simple procedure, combined with the Expected Utility theorem. We know that, if your preferences are in accordance with this theory, your choice is driven by expected utility. We will consider risky choices which have a particularly simple form: two outcomes are possible and they are equally likely. We will denote such a risky choice by \((x,y)\) where the two outcomes are £x and £y. So \((x,y)\) denotes a risky prospect in which the outcome could be £x with probability \(\frac{1}{2}\) or £y with probability \(\frac{1}{2}\).

Consider the risky choice \((0,100)\) – there is a 50-50 chance you will get £100 and a 50-50 chance you get £0. It is the simple game that I auction off. You should ask yourself: at what price in this auction would you drop out? £10? £20? £23? At what price? This is a crucial introspection. It is not easy to do, but it determines your reservation price for playing the little game with me. At this reservation price you are indifferent between paying that price and playing the game with me, or not playing the game.

---

6 Of course, in addition to your present wealth - something we will take for given from now on.
Let us denote this reservation price by $x_1$. Notice crucially that, in general, it varies from individual to individual. For you, you are indifferent between £$x_1$ for sure (for that is what you are giving up if you play the game with me) and the 50-50 risky gamble between £0 and £100. It follows therefore that the expected utility of £$x_1$ for sure must be equal for you to the expected utility of the 50-50 risky gamble between £0 and £100. The first of these (the expected utility of £$x_1$ for sure) is tautologically equal to the utility of £$x_1$, while the second of these (the expected utility of the 50-50 risky gamble between £0 and £100) is equal to $\frac{1}{2} u(£0) + \frac{1}{2} u(£100)$, which is equal to 0.5 by virtue of the equation above. It immediately follows that

$$u(£x_1) = 0.5$$

We have therefore found a third point on your utility function. Insert it in your graph: the amount of money £$x_1$ for which you are indifferent between it and (0, 100) gives you utility 0.5.

To find other points we simply repeat the procedure. For example, to find the amount of money which gives you utility 0.25, we start with a gamble which we know gives you expected utility 0.25 and ask you to tell us the amount of money which, received with certainty, makes you indifferent with that gamble. What gamble do we know gives you expected utility 0.25? There are two that we know of – one is the 50-50 gamble between £0 and £$x_1$. Perhaps you would like to think what the other is. So one way to proceed is to ask you: what is amount of money £$x_2$ which makes you indifferent between that amount of money and the risky choice (0, $x_1$)? The answer £$x_2$ is such that $u(£x_2) = 0.25$. This gives you a fourth point on your utility function. Other points can be found in the same way. You should check out the following.

If you are indifferent between £$x_1$ and (0, 100) then $u(£x_1) = 0.5$
If you are indifferent between £$x_2$ and (0, $x_1$) then $u(£x_2) = 0.25$
If you are indifferent between £$x_3$ and ($x_1$, 100) then $u(£x_3) = 0.75$
If you are indifferent between £$x_4$ and (0, $x_2$) then $u(£x_4) = 0.125$

Clearly you can continue this way indefinitely and hence build up a more and more accurate picture of your utility function. You should do this, along with your fellow-students, and then compare the functions that you have obtained. If your function is linear everywhere, you are risk-neutral everywhere. If it is concave everywhere, you are risk-averse everywhere – the more concave, the more risk-averse. If it is convex everywhere, you are risk-loving everywhere – the more convex, the more risk-loving. Of course, you could have a utility function which is concave in some parts and convex in others – indicating that for certain gambles you are risk-averse whereas for other gambles you are risk-loving.

### 24.12 Mathematical Appendix

First we provide a sketch of the Expected Utility theorem. We should emphasise that it is only a sketch – and many of the steps are omitted. Knowledge of the proof of this theorem is not expected from students following this course.

This theorem is based on axioms of rational behaviour. If the axioms are true then so is the theorem. We consider risky gambles whose final outcome is one from a set $A_1, A_2, ..., A_I$ of payoffs. We first assume that the individual whose preferences we are describing can rank these final payoffs from best to worst and can therefore define the best and the worst. Number the payoffs so that $A_1$ is the best (most preferred by the individual) and $A_I$ the worst (least preferred). The first axiom (Continuity) is that for all $A_i$ there is some probability $u_i$ for which the individual is indifferent
between $A_i$ and a gamble between the best $A_i$ and the worst $A_i$ with respective probabilities $u_i$ and $1-u_i$. Note, rather trivially, that $u_i$ must be 1 and $u_i$ must be 0. Now define the utility of outcome $A_i$ as $u_i$.

The second axiom (Dominance) says that in any two gambles involving only the best and the worst outcomes, the gamble which has the highest probability of the best outcome (and hence the lowest probability of the worst outcome) is preferred to the other. These first two axioms seem harmless.

We now come to the key axiom, that we have already discussed in the text: the Independence Axiom. Let us express it in a slightly different (but equivalent) form. Let $C$ and $D$ be risky choices and suppose we have an individual who says that he or she is indifferent between $C$ and $D$. Let $E$ be any other risky choice. The Independence Axiom then states that he or she should also be indifferent between the gamble (call it $G$) which yields $C$ with probability $p$ and $E$ with probability $(1-p)$ and another gamble (call it $H$) which yields $D$ with probability $p$ and $E$ with probability $(1-p)$. This is a more contentious axiom. It is subject to some criticism, but it is needed for the proof of the Expected Utility theorem. This we now show.

Consider two general gambles: $C$ which yields outcomes $A_1, A_2, ..., A_I$ with probabilities $p_1, p_2, ..., p_I$ and $D$ which yields outcomes $A_1, A_2, ..., A_I$ with probabilities $q_1, q_2, ..., q_I$. Using the Independence Axiom $I$ times with each of $C$ and $D$, we can argue that the individual should be indifferent between $C$ and the two-stage gamble in which the outcomes $A_1, A_2, ..., A_I$ are replaced by the gambles between the best and the worst with which they said they were indifferent with the continuity axiom. We can argue the same for $D$. Now we invoke the Reduction of Compound Gambles Axiom which states that the individual should be indifferent between a two-stage gamble and the equivalent single-stage gamble which results using the reduction of compound probabilities rule. This implies that the individual is indifferent between $C$ and the gamble between the best and the worst with respective probabilities $p_1u_1 + p_2u_2 + ... + p_Iu_I$ and $1 - (p_1u_1 + p_2u_2 + ... + p_Iu_I)$. Similarly he or she should be indifferent between $D$ and the gamble between the best and the worst with respective probabilities $q_1u_1 + q_2u_2 + ... + q_Iu_I$ and $1 - (q_1u_1 + q_2u_2 + ... + q_Iu_I)$.

Now examine these final two gambles – both involving just the best and worst outcomes. Which should the individual prefer? Using the Dominance axiom, that with the highest probability of getting the best outcome. So we get the result that

$C$ is preferred to $D$ if and only if $p_1u_1 + p_2u_2 + ... + p_Iu_I$ is greater than $q_1u_1 + q_2u_2 + ... + q_Iu_I$.

Now note that $p_1u_1 + p_2u_2 + ... + p_Iu_I$ is the Expected Utility from the gamble $C$ and $q_1u_1 + q_2u_2 + ... + q_Iu_I$ is the Expected Utility from the gamble $D$ because, using the continuity axiom, we have defined $u_1, u_2, ..., u_I$ as the respective utilities of the outcomes $A_1, A_2, ..., A_I$. Thus we have the Expected Utility theorem which says that choice between risky gambles should be determined by the expected utility of the gambles: the higher the expected utility the more preferred the gamble.

Second we provide a derivation of the slope of an indifference curve implied by the Expected Utility Model.

An indifference curve in $(c_1, c_2)$ space is given, as ever, by

$$U(c_1, c_2) = \text{constant}$$
If we substitute in the specification of the Expected Utility Model from (24.1) we get the following equation for an indifference curve in \((c_1, c_2)\) space.

\[ \pi_1 u(c_1) + \pi_2 u(c_2) = \text{constant} \]

To find the slope of the indifference curve we differentiate this totally, thus getting

\[ \pi_1 \frac{du(c_1)}{dc_1} dc_1 + \pi_2 \frac{du(c_2)}{dc_2} dc_2 = 0 \]

where \(u'(c)\) denotes the derivative of \(u(c)\) with respect to \(c\). From this we get the slope of an indifference curve

the slope of an indifference curve is \(- \frac{\pi_1 du(c_1)/dc_1}{\pi_2 du(c_2)/dc_2}\)

which is equation (24.2)