Chapter 23: Choice under Risk

23.1: Introduction

We consider in this chapter optimal behaviour in conditions of risk. By this we mean that, when the individual takes a decision, he or she does not know with certainty what might happen. However we assume that the individual can list the various possible things that may happen and can attach probabilities to the various possibilities. For simplicity we assume that the set of possible eventualities contains only\(^1\) two possibilities - or 'states of the world' - which we call state 1 and state 2. We denote by \(\pi_1\) and \(\pi_2\) the respective probabilities attached to these two states and we assume that the decision maker knows these probabilities. Of course \(\pi_1 + \pi_2 = 1\). So state 1 happens with probability \(\pi_1\) and state 2 happens with probability \(\pi_2\). Further we assume that the income that the individual receives depends upon the state of the world that occurs: we use \(m_1\) to denote the income the individual receives if state 1 occurs and \(m_2\) the income if state 2 occurs. \textit{Ex ante}, before the uncertainty is resolved the individual does not know which state will occur; \textit{ex post}, one and only one of the two states will occur and the individual will receive \textit{either} an income of \(m_1\) \textit{or} an income of \(m_2\). For simplicity in this chapter we will assume that the individual then consumes his or her income – and thus gets utility directly from this income.

Now it may be the case that the individual is happy with this initial position - that is, is happy consuming \(m_1\) if state 1 occurs and \(m_2\) if state 2 occurs. But note that this is a position of \textit{ex ante} risk if \(m_1\) and \(m_2\) are different. The individual might not like such a risky future. He or she may prefer an alternative scenario. For example, if the individual really dislikes risk, he or she would want to re-arrange things so that he or she received the same income whichever state occurred. The way that the individual may be able to re-arrange the possibilities is through insurance. An insurance market specifically exists to re-arrange risk. For example, if you are worried that you may have an accident that would reduce your income, you might take out insurance so that if the accident occurred the insurance company would pay you money in compensation. This means that you pay money (the insurance premium) to the insurance company if the accident does not happen, and the company pays money to you if the accident does happen. We can think of this being done by buying and selling \textit{contingent income} - income contingent on which state occurs. So a typical insurance contract involves you agreeing to give money to the insurance company if the accident does not occur and the company agreeing to give you money if the accident does occur. Usually there is a relationship between the amount you give to the company if the accident does not occur and the amount that they give you if the accident does occur. This relationship is obviously affected by the probability of the accident occurring. We shall study this in more detail shortly.

23.2: The Budget Constraint

We define contingent income in a particular state of the world as the income you would get or pay if that state of the world occurred. It is a good or a commodity like any other except that it is received or paid only if a particular state occurs. Let us denote the price of state 1 contingent income by \(p_1\) and the price of state 2 contingent income by \(p_2\). This means that for each unit of state 1 income that you buy the cost is \(p_1\) and for each unit of state 2 income that you buy the cost is \(p_2\). Similarly, for each unit of state 1 income that you sell your revenue is \(p_1\) and for each unit of state 2 income that you sell your revenue is \(p_2\). You buy and sell state contingent income \textit{before} you know

\(^1\) All the material can be generalised to more than 2 possibilities.
whether state 1 or state 2 will occur. Then Nature decides which state is to occur and then you have the payoffs. **If state 1 occurs** then for each unit of state 1 contingent income that you have bought you receive a payment of 1 unit of money; for each unit of state 1 contingent income that you have sold you have to pay 1 unit of money. **If state 2 occurs** then for each unit of state 2 contingent income that you have bought you receive a payment of 1 unit of money; for each unit of state 2 contingent income that you have sold you have to pay 1 unit of money. Note carefully the sequence of events: ex ante you do not know whether state 1 or state 2 will occur (though you do know the respective probabilities); ex ante you have to decide how much state contingent income to buy or sell; ex post you are told which state has occurred and then you receive or pay money depending which state occurs and whether you have bought or sold income contingent on that state. For example, suppose ex ante you have bought 5 units of state 1 contingent income (at a cost of 5\(p_1\)) and you have sold 10 units of state 2 contingent income (receiving a payment of 10\(p_2\)) then ex post if state 1 occurs you get paid 5 units of money whereas if state 2 occurs you have to pay 10 units of money.

With state contingent income you can change the riskiness of your position. Suppose, for example, that \(m_1 = 40\) and \(m_2 = 60\), that is, that your ex ante income if state 1 were to occur is 40 and your ex ante income if state 2 were to occur is 60 (so you would be better off ex post if state 2 were to occur and worse off if state 1 were to occur). You could convert this into a position of ex ante certainty through state contingent income. Suppose for simplicity that \(p_1 = p_2\). Then you simply buy 10 units of state 1 contingent income and sell 10 units of state 2 contingent income (the cost of the purchase of state 1 contingent income exactly equalling the revenue from the sale of state 2 contingent income). Then if state 1 were to occur your income would be 40 (your ex ante income in that state) plus 10 (the income from your ownership of 10 units of state 1 contingent income) which is 50. If state 2 were to occur your income would be 60 (your ex ante income in that state) minus 10 (the payment you have to make on the 10 units of state 2 contingent income that you have sold) which is 50. So – regardless of which state occurs – you always end up with an income of 50.

What is your budget constraint in this world? Suppose you start with income \(m_1\) if state 1 occurs and income \(m_2\) if state 2 occurs. Suppose you want to re-arrange things in such a way that you would have consumption \(c_1\) if state 1 occurred and consumption \(c_2\) if state 2 occurred. What constraints are you under at prices \(p_1\) and \(p_2\) for state contingent income? Simply the usual condition:

\[
p_1c_1 + p_2c_2 = p_1m_1 + p_2m_2 \tag{23.1}\]

Why? Well suppose that you feel that your income if state 1 were to occur would be too low - and therefore that you want to re-arrange things so that your consumption if state 1 were to occur would be \(c_1\) where \(m_1 < c_1\). Then you would have to buy \(c_1 - m_1\) units of state 1 contingent income at a cost of \(p_1 (c_1 - m_1)\) and to finance this purchase you would have to sell a sufficient number of units of state 2 contingent income to pay for this. If you sell \(m_2 - c_2\) units it would yield \(p_2(m_2 - c_2)\) in revenue. So you would have to have \(p_2(m_2 - c_2)\) equal to \(p_1(c_1 - m_1)\). Which gives us \(p_2(m_2 - c_2) = p_1(c_1 - m_1)\) which yields the budget constraint written above. Alternatively suppose that you feel that your income if state 2 were to occur would be too low - and therefore that you want to re-arrange things so that your consumption if state 2 were to occur would be \(c_2\) where \(m_2 < c_2\). Then you would have to buy \(c_2 - m_2\) units of state 2 contingent income at a cost of \(p_2(c_2 - m_2)\) and to finance this purchase you would have to sell a sufficient number of units of state 1 contingent income to pay for this. If you sell \(m_1 - c_1\) units it would yield \(p_1(m_1 - c_1)\) in revenue. So you would have to have \(p_1(m_1 - c_1)\) equal to \(p_2(c_2 - m_2)\). Which gives us \(p_1(m_1 - c_1) = p_2(c_2 - m_2)\) which yields the budget constraint written above. Notice that plotted in \((c_1, c_2)\) space this is a straight line with slope \(-p_1/p_2\) which passes through the initial income point \((m_1, m_2)\). This all looks very familiar.
Indeed it is all very familiar – state contingent income is just a commodity like any other except that it yields a payoff in just one state of the world.

23.3: A Fair Insurance Market

An insurance market is simply a market for buying and selling state contingent income. The prices of state contingent income obviously have some connection with the probability of that state occurring. For example if state 1 is very likely to occur then the price of state 1 contingent income should be high as there is a high chance of a payoff. Conversely if state 1 is very unlikely to occur then the price of state 1 contingent income should be low as there is a low chance of a payoff.

Let us see what the price should be in a particular type of insurance market – one we term a fair insurance market. What we mean by this is that, whether you are buying or selling state contingent income, you expect to break even on average. Some of the time you are paying out money, some of the time you are receiving money. It is fair if, on average, the amount you pay out is equal to the amount you receive.

Consider state 1 contingent income. A unit of this costs \( p_1 \). If state 1 actually does occur each unit brings in 1 unit of money. If state 1 does not occur each unit brings in nothing. These two events happen with probability \( \pi_1 \) and \( \pi_2 \) respectively. Thus a proportion \( \pi_1 \) of the time 1 unit of state 1 contingent income brings in 1 unit of money while a proportion \( \pi_2 \) of the time it brings in nothing. On average it brings in \( \pi_1 \times 1 + \pi_2 \times 0 = \pi_1 \) in money. For it to be fair this should equal the price of state 1 contingent income – which gives us the condition for fair state 1 contingent income or for fair insurance:

\[
p_1 = \pi_1
\]

Similarly for state 2 contingent income:

\[
p_2 = \pi_2
\]

So for it to be a fair insurance market the price for 1 unit of state contingent income should be equal to the probability of that state occurring. So for a state that happens with probability \( \frac{1}{2} \), for example, the fair price for 1 unit of income contingent on that state should be \( \frac{1}{2} \): \( \frac{1}{2} \) the time it pays out 1, \( \frac{1}{2} \) the time it pays out nothing; on average it pays out \( \frac{1}{2} \) - hence its fair price.

The implication of a fair insurance market is that the insurance company breaks even on average – the premiums it gets in are exactly equal to the claims it pays out. You may regard this as a bit unrealistic when you look at the profits that insurance companies make in practice. But you should note that a large part of an insurance companies business is in investing the premiums it gets in – remember that it gets in the premiums before it pays out any claims, and in the meantime it invests the premiums. Indeed it is not far from the truth to say that insurance companies make virtually all of their profits from investment and virtually none from the insurance itself. If the market for insurance is fair, then indeed they make no profits from the insurance.

One implication of a fair market – in which insurance companies make zero profits from insurance itself – is that the insurance business is a simple redistribution: those people who are lucky, in that they do not need to make any claims, pay through their premiums for those people who are unlucky and need to claim. Of course, individuals could do some of this re-distribution themselves through
time, but most prefer the insurance market to do it for them. The insurance market simply redistributes the risk.

We present in figure 23.1 an example in which the two states of the world are equally likely and in which we have fair insurance so that $p_1 = p_2 = 0.5$. In this example the individual starts off with an income of 30 if state 1 occurs and of 50 if state 2 occurs. The ‘X’ indicates the *ex ante* endowment point – note that the budget constraint must pass through this point. The budget constraint in this example has slope $-1$. More generally it has slope $-\frac{p_1}{p_2}$ - or in a fair market - $\pi_1/\pi_2$ where $\pi_1$ and $\pi_2$ are the probabilities of state 1 and 2 respectively.

![Diagram](image)

**23.4: Preferences**

Now let us think about preferences. First, one very important thing should be noticed: the space in which we are working should be carefully noted. On the horizontal axis is the income/consumption (we are assuming that these are the same) that the individual would have if state 1 occurred and on the vertical axis the income/consumption that the individual would have if state 2 occurred. *Ex ante* the individual does not know which will occur but *ex post* only one will occur. The individual will only get one of the two incomes - not both. This is a crucial difference compared to all our previous analyses. In all these previous analyses the individual received both of the goods on the two axes; in this chapter (and the next two) the individual receives only one of them. He or she has to decide in advance - before he or she knows which state will occur - a point in this space. Then Nature decides whether he or she gets the amount on the vertical axis or the amount on the horizontal axis. Someone who does not like *ex ante* risk will try and make the two possible incomes as close to each other as possible, so that whatever happens the income received is almost the same. Someone who likes risk may like to gamble a bit. But, whatever the attitude to risk we can represent the preferences (towards risk) of the individual in this quantity space in the normal way by using indifference curves. One thing we can say right away is that, if the individual is risk averse – that is, he or she has an aversion to risk (a concept we shall define more carefully shortly) - then his or her indifference curves must be convex. Why? Because the individual will want more and more compensation in terms of state 2 income for giving up units of state 1 income. Also we can say that the shape of the indifference curves will depend upon the two probabilities - if the probability of state 2 occurring is low, for example, the individual would require more compensation in terms of state 2 income for giving up state 1 income than if the probability of state 2 occurring is high. We will give some more specific examples in the next chapter, but let us give an example here.
We start with the case where the individual is risk-averse and in which both probabilities are equal to 0.5, that is the two states are equally likely. It seems reasonable in this case that the preferences are symmetric as in figure 23.3.

![Figure 23.3: A risk-averse individual when the probabilities are 0.5 and 0.5](image)

In this figure the 45 degree line is the certainty line - along this the individual would receive the same income whichever state of the world occurred.

We have argued that if the individual is risk-averse then he or she will have convex indifference curves in this space. Working the other way we see from the figure above for example that this individual prefers the point (50, 50) to the points (25, 75) and (0, 100) – despite the fact that all three of these points have the same expected income – namely 50 (recall that the two states are equally likely). So the individual with the above preferences strictly prefers the certainty of 50 to any risky prospect with expected income 50. This is presumably what we mean by a risk-averse person.

What about a risk-neutral person? By this we presumably mean someone who simply ignores the risk – it does not affect his or her preferences. Such a person cares only about the expected income from some point and does not worry about the risk. Can we specify his or her indifference curves? Well, given some point \( (c_1, c_2) \) in this space the expected consumption/income is

\[ \pi_1 c_1 + \pi_2 c_2 \]

if the respective probabilities are \( \pi_1 \) and \( \pi_2 \), because a proportion \( \pi_1 \) of the time the consumption/income is \( c_1 \) and a proportion \( \pi_2 \) of the time the consumption/income is \( c_2 \). For a risk-neutral individual his or her utility, as we have just argued, is determined solely by the expected consumption/income, so an indifference curve for him or her is given by

\[ \pi_1 c_1 + \pi_2 c_2 = \text{constant} \]

This indifference curve is a straight line in \( (c_1, c_2) \) space with slope \( -\pi_1/\pi_2 \). In the case of equally-likely states of the world we get the indifference map for a risk-neutral individual shown in figure 23.5.
For a risk-neutral individual when $\pi_1 = 0.4$ and $\pi_2 = 0.6$ we have figure 23.6. Note that the slope depends upon the probabilities.

![Diagram 23.5: a risk neutral individual when the probabilities are 0.5 and 0.5](image1)

![Diagram 23.6: a risk neutral individual when the probabilities are 0.4 and 0.6](image2)

You should now be able to work out that someone who likes risk – a risk-lover – will have indifference curves that are **concave** – so, for example, with equal probabilities, they will strictly prefer $(100, 0)$ to $(50, 50)$ even though these two points have the same expected income $(50)$. So we get the nice result that

In $(c_1, c_2)$ space the indifference curves are convex, linear or concave according as to whether the individual is risk-averse, risk-neutral or risk-loving.

### 23.5: Optimal Choice

Apart from the interpretation, the analysis that we are doing should be remarkably familiar. Indeed we can find the optimal decision and see how it changes when the prices and the incomes change in the usual way. There is one result that is of particular interest – and one that we shall explore in more detail and in more generality in chapter 24 – and that concerns the amount of insurance that a risk-averse person buys. We shall consider here only the symmetric case, as I need to appeal to your intuition, but we shall consider the more general case (under a specific assumption about preferences) in chapter 24. So I assume in what follows in this section that the two states are equally likely and thus both have probabilities $\frac{1}{2}$.

It was asserted above that in the case of equal probabilities – both states equally likely – then the indifference map should be symmetrical, and therefore in the case of a risk-averter should look like figure 23.3 above. The reason why intuition suggests symmetry is simply that there is no reason to treat the two states of the world any differently, given that they are equally likely. One particular
implication of the assumption of symmetry concerns the slope of the indifference curves along the line $c_1 = c_2$, which is drawn in on the figure. This line we rather naturally call the *certainty line* – along it the consumption/income of the individual is the same irrespective of which state of the world occurs. If the individual chooses to locate him or herself along that line, then we can say that the individual has chosen a position of *ex ante* certainty. Or, in other words, that he or she has chosen to be completely insured in the sense that the actual state of the world is irrelevant. Given our assumption of the symmetry of the indifference map it follows that the slope of each indifference curve at any point on the certainty line must equal $-1$.

Let us now consider the optimal insurance decision of this individual *if faced with fair insurance*. This means that the prices are equal to the probabilities and hence are both equal to $\frac{1}{2}$. This means that the budget constraint has slope $-\frac{1}{2}/\frac{1}{2} = -1$. We illustrate this in figure 23.7.

23.7: optimal choice for a risk averse individual (0.5 and 0.5) with fair insurance

The initial point is (30, 50) – without insurance the individual gets consumption/income equal to 30 if state 1 occurs and gets consumption/income equal to 50 if state 2 occurs. The optimal point is, as ever, the point on the budget constraint (the line through the original point X) on the highest possible indifference curve. We have indicated it with an asterisk.

We see that the optimal point is on the certainty line! (It is there because the slope of the budget constraint is $-1$ which is equal to the slope of the indifference curves along the certainty line.) So the individual chooses to be fully insured. *Ex ante*, the individual buys 10 units of state 1 contingent income and sells 10 units of state 2 contingent income and therefore moves from (30, 50) to (40, 40). Whatever state of the world occurs he or she will get consumption/income of 40: if state 1 occurs he or she gets the income of 30 plus the income from the 10 units of state 1 contingent income bought; if state 2 occurs he or she gets the income of 50 minus the cost of the 10 units of state 2 contingent income sold. If you like you can think of this as an insurance against state 1 – the state in which the individual gets a low income. If this bad (because he or she gets a low income) state occurs the insurance pays 10; if the state does not occur the individual pays 10 to the company.

Is this surprising? In a sense not – because we have assumed that the individual is risk averse and we have assumed fair insurance. And what does fair insurance do? It offers the chance to reduce the riskiness of future income without changing the expected income. And a risk-averse person – effectively by definition – will always prefer a less risky to a more risky prospect with the same expected value. But the result nevertheless is interesting.

The above assumed fair insurance. What happens if we do not have fair insurance? Suppose that we have *more-than-fair* insurance in the sense that $p_1 < \pi_1$. See figure 23.7 below in which we have put $p_1 = 0.2$. What does the individual do?
The individual moves from the initial point to the asterisked point (77, 31) – buying some 47 units of state 1 contingent income and selling some 19 units of state 2 contingent income. Note that in the new position the individual has an expected income of 54 – considerably above the initial expected income of 40. So this individual, despite being risk-averse chooses to bet on state 1 happening and increases his expected income – because he or she has been offered more-than-fair insurance. You might like to reflect what he or she would do if offered less-than-fair insurance for state 1 – that is a price $p_1 > \pi_1$.

We shall explore more of the implications of insurance markets in chapter 24 where we introduce a particular model of preferences under risk. In the meantime you could consider what happens to the optimal decision if $p_2$ changes or either of the ex ante incomes $m_1$ and $m_2$.

**23.6: Expected Values**

We should perhaps formalise some definitions. If a variable $X$ takes values $x_1$ and $x_2$ with probabilities $\pi_1$ and $\pi_2$ respectively then the expected value of $X$ is given by

$$\text{Expected value of } X = \pi_1 x_1 + \pi_2 x_2$$  \hspace{1cm} (23.2)

This is just a weighted average of the possible values of $X$ weighted by the probabilities. It is called an expected value because if we were to observe $X$ a large number of times and calculated the average then this is what we would expect it to be.

We can generalise: if a variable $X$ takes values $x_1, x_2, ..., x_i, ..., x_I$ with probabilities $\pi_1, \pi_2, ..., \pi_i, ..., \pi_I$ respectively then the expected value of $X$ is given by

$$\text{Expected value of } X = \pi_1 x_1 + \pi_2 x_2 + \ldots + \pi_i x_i + \ldots + \pi_I x_I$$  \hspace{1cm} (23.3)

Once again we take a weighted average of the possible values of $X$ weighted by the probabilities.

**23.7: Unfair Insurance**

We have assumed above what we have called a perfect or fair insurance market. This is where $p_1 = \pi_1$ and $p_2 = \pi_2$. An unfair market is where if you want to buy state 1 income then the price is too high: $p_1 > \pi_1$; and where if you want to buy state 2 income its price is too high: $p_2 > \pi_2$. Notice what this does - it induces a kink in the budget constraint at the initial endowment point. Does this remind you of something?
23.8: Summary

We have looked at choice under risk – where individuals do not know ex ante what is going to happen but can attach probabilities to the various possibilities.

We have argued that an insurance market enables the buying and selling of risk, and it does this through the buying and selling of state contingent income. One unit of income conditional on some state pays 1 unit of money if that state occurs and pays nothing otherwise.

*We saw that in a fair insurance market the price of state contingent income should equal the probability of that state occurring.*

As far as preferences were concerned *we showed that an individual has convex, linear or concave indifference curves according to whether he or she is risk-averse, -neutral or –loving.*