Chapter 6: Supply and Demand with Income in the Form of Endowments

6.1: Introduction

This chapter and the next contain almost identical analyses – concerning the supply and demand implied by different kinds of preferences - though they differ in one crucial respect. In this chapter it is assumed that the individual receives an income in the form of an endowment (of the two goods considered in the analysis); in the next chapter it is assumed that the individual receives an income in the form of money. Otherwise the analyses are almost identical. The reasons why there are separate chapters are two-fold: first, the material of this chapter follows naturally the discussion contained in chapters 3 and 4, in which the individual’s income was in the form of endowments; second, the case discussed in this chapter is, in a sense that may not become obvious until later, a more general case which can be applied elsewhere. In fact, it is used repeatedly throughout the book, and in particular in Part 3.

It follows therefore that chapter 7 is in one sense a special case of this chapter – so when you have mastered this chapter it will be easier for you to master chapter 7.

In this chapter it is assumed that the individual has an income in the form of an endowment of each of the two goods. By this we mean that at the start of the day, the individual has in his or her possession a quantity of each of the two goods (though the quantity of one or the other could be zero). There is no money in this chapter – the only thing that the individual can do is to exchange one of the goods that he or she possesses for more of the other. This is a very useful framework to use and we shall exploit it often in the chapters that follow. Given this starting point, the chapter then explores, for different kinds of preferences, what is the best thing for the individual to do at any given prices – should he or she buy good 1 and sell good 2, should he or she sell good 1 and buy good 2, should he or she do nothing? And if he or she should buy or sell, how much?

We then explore how these optimal demands and supplies depend upon the key exogenous variables. By the expression ‘exogenous variable’ we mean a variable that is outside the control of the individual – that he or she has to take as given. In this instance the key exogenous variables are the prices of the two goods and the endowments of the two goods. We examine how demand and supply depend on these variables. These kind of exercises are called by economists comparative static exercises. In the particular case when we find the relationship between the demand or supply of some good and the price of that good itself, such a relationship is usually called the demand or supply curve of that good (though we should note that this terminology is somewhat misleading: in general the demand or supply of some good depends upon both the price of that good but also the price of the other good as well as the endowments of the two goods).

In this chapter we look at demand and supply for a variety of different preferences. Some results you should remember (these will be pointed out to you) but it is more important that you understand the general methodology that is used. After reading this chapter you should be able to apply this methodology to other kinds of preferences – at least in principle - though you might find the mathematical detail a little difficult.
6.2: The Budget Constraint with Income in the Form of Endowments

We continue to use the framework adopted in chapter 5. We are considering an individual’s preferences and choice over two goods, good 1 and good 2. We put the quantity of good 1, which I denote by \( q_1 \), on the horizontal axis and the quantity of good 2, which we denote by \( q_2 \), on the vertical axis. We use \( p_1 \) and \( p_2 \) to denote the respective prices of the two goods.

In this chapter we assume that the individual gets his or her income in the form of endowments of the two goods – a quantity \( e_1 \) (which could be zero) of good 1 and a quantity \( e_2 \) (which could be zero) of good 2. (Obviously one or other of these two endowments must be non-zero otherwise there is nothing to discuss.)

We presume that the prices of the two goods are exogenous to the individual – he or she simply takes them as given. Given any prices the individual has various trading opportunities – as defined by his or her budget constraint. What form does this take? Well, obviously it must pass through the initial endowment point as the individual can always choose not to do anything. What else? Well we can argue that the budget constraint must be such that the cost of the bundle consumed is equal to the value of the initial endowment. That is,

\[
p_1 q_1 + p_2 q_2 = p_1 e_1 + p_2 e_2 \quad (6.1)
\]

The left-hand side of this is simply the cost of the consumed bundle; the right-hand side is the value of the initial endowment. This is the individual’s budget constraint. We note that in \((q_1, q_2)\) space this is simply a straight-line passing through the initial point \((e_1, e_2)\) and having slope \(-p_1/p_2\).

There are other ways we can write this budget constraint – which might be more understandable if we view the individual as starting at the point \((e_1, e_2)\) and moving from there to \((q_1, q_2)\) by either buying good 1 and selling good 2 or selling good 1 and buying good 2.

If the individual chooses to buy (more of) good 1 then \( q_1 \) must be bigger than \( e_1 \) and \( q_2 \) smaller than \( e_2 \). Let us therefore write (6.1) in the form:

\[
p_1 (q_1 - e_1) = p_2 (e_2 - q_2) \quad (6.2)
\]

The left-hand side of this is simply the cost of buying the extra units of good 1. This purchase has to be financed. It is financed by selling \((e_2 - q_2)\) units of good 2, which raises enough in revenue to finance the purchase.

Alternatively the individual chooses to sell some of good 1 and buy some more of good 2. In this case \( q_1 \) is less than \( e_1 \) and \( q_2 \) is greater than \( e_2 \). Let us re-write (6.1) in the form:

\[
p_1 (e_1 - q_1) = p_2 (q_2 - e_2) \quad (6.3)
\]

In this the left-hand side is the money raised by selling \((e_1 - q_1)\) units of good 1 at a price of \( p_1 \) while the right-hand side is the cost of financing the purchase of \((q_2 - e_2)\) extra units of good 2 at a price of \( p_2 \). These two expressions must be equal.

Equations (6.1), (6.2) and (6.3) of course are all the same. They each represent a straight line in \((q_1,q_2)\) space (the equations are linear in \( q_1 \) and \( q_2 \)) passing through the endowment point \(X\) (substitute \( q_1 = e_1 \) and \( q_2 = e_2 \) in the equation and see that these values satisfy the equation) with slope equal to \(-p_1/p_2\) (the price of good 1 relative to the price of good 2). To see this latter point
write the equation as \( q_2 = \frac{(p_1 e_1 + p_2 e_2)/p_2 - (p_1/p_2) q_1}{p_2} \) - an equation linear in \( q_2 \) and \( q_1 \) with the coefficient on \( q_1 \) equal to \(-p_1/p_2\).

You should note very carefully that the budget line must pass through the initial endowment point - the individual can always choose to do nothing and stay at that initial point. This fact has important consequences – not least of which is the budget constraint rotates around the initial point when either \( p_1 \) or \( p_2 \) change. More precisely, when either \( p_1 \) rises or \( p_2 \) falls, the magnitude of the slope increases so that the budget line rotates clockwise around the original point. Note that if the individual is buying good 1 and selling good 2 such a rotation makes the individual worse off (because the price of the good that he or she is buying goes up or the price of the good that he or she is selling goes down), whereas if the individual is selling good 1 and buying good 2 such a rotation makes the individual better off (because the price of the good that he or she is selling goes up or the price of the good that he or she is buying goes down). We shall explore such cases in what follows.

What is optimal for the individual to do, given any endowment and any prices, depends upon the preferences of the individual. In what follows we shall take different preferences and work out the implications. In particular, we shall consider Cobb-Douglas preferences, Stone-Geary preferences, then perfect substitutes then perfect complements. What you should get out of this is the idea that the preferences determine the demands.

6.3: Choice with Cobb-Douglas Preferences

While much of our analysis will be graphical we will find it useful to start this section with a little mathematics. This can be ignored if you do not like mathematics – as we shall later verify the results with particular numerical examples. In any case the results shall be explained to you – what is more important is that you understand the explanation and that you appreciate that these particular demands are a consequence of these particular preferences.

You will recall that Cobb-Douglas preferences can be represented by the utility function

\[
U(q_1, q_2) = q_1^a q_2^{1-a}
\]

or by the function

\[
U(q_1, q_2) = a \ln(q_1) + (1-a) \ln (q_2) \quad (6.4)
\]

For any given budget constraint the individual wants to find the point on the constraint that is on the highest possible indifference curve – that is, where the utility is maximised. The budget constraint is given by (6.1) the utility function by (6.4). So the individual’s problem is to find the point \((q_1, q_2)\) which maximises \(a \ln(q_1) + (1-a) \ln (q_2)\) subject to the constraint that \(p_1 q_1 + p_2 q_2 = p_1 e_1 + p_2 e_2\).

Now there are various ways that one can solve this constrained maximisation problem. One way is provided in the Mathematical Appendix to this chapter. As will be seen there, the solution to the problem of the maximisation of \(U(q_1, q_2)\) subject to the budget constraint \(p_1 q_1 + p_2 q_2 = p_1 e_1 + p_2 e_2\) is given by the equations

\[
q_1 = a(p_1 e_1 + p_2 e_2)/p_1 \quad \text{and} \quad q_2 = (1-a)(p_1 e_1 + p_2 e_2)/p_2 \quad (6.5)
\]

These expressions are the demand functions for the two goods. They are vitally important expressions – you should remember them. Actually they are probably easier to remember in the following form:
Note that \((p_1 e_1 + p_2 e_2)\) is simply the value of the initial endowment – the individual’s income if you like. So the first of these two equations (6.6) says that the amount that the individual spends on good 1 \((p_1 q_1)\) is simply a constant fraction \(a\) of his or her income, while the second equation says that the amount that the individual spends on good 2 \((p_2 q_2)\) is a constant fraction \((1-a)\) of his or her income.

So the story with Cobb-Douglas preferences is clear and simple: the individual takes his or her income \((p_1 e_1 + p_2 e_2)\) and divides it up into two unequal (unless \(a = 0.5\)) parts. He or she spends a fraction \(a\) on good 1 and a fraction \((1-a)\) on good 2. Hence the amount spent on each good is a constant fraction of the income. This implies that the quantity purchased depends upon the price of the good.

So if the value of \(a\) is 0.25 then the individual spends one-quarter of his or her income on good 1 and a three-quarters of his or her income on good 2. So if the value of his income is £100 then £25 is spent on good 1 and £75 on good 2. The actual quantity purchased depends upon the price of the good. So if he or she spends £25 on good 1 and its price is £5 per unit then he or she buys 5 units, if the price is £2 per unit then he or she buys 12.5 units. And so on.

There is one complication that may cause confusion. When income is in the form of endowments then the value of the income is affected by the prices. So a price change affects both the value of the income and also the purchasing power of any given sum of money. For example, let us continue to work with \(a = 0.25\). Take \(e_1 = 15\) and \(e_2 = 25\). At prices \(p_1 = p_2 = £1\) the value of this endowment is £40, and our Cobb-Douglas individual with \(a=0.25\) spends £10 of this (a quarter) on good 1 and £30 (three-quarters) on good 2. At prices \(p_1 = p_2 = £1\) this means he or she can buy 10 units of good 1 and 30 units of good 2. Now suppose that the price of good 2 rises to \(p_2 = £2\) while the price of good 1 remains unchanged. Then the value of the endowment (the income of the individual) rises to £65 (= 15 x £1 + 25 x £2). The Cobb-Douglas individual continues to spend a quarter on good 1 and three-quarters on good 2. So he or she spends £16.25 on good 1 and £48.75 on good 2. With the £16.25 he or she can buy 16.25 units of good 1 while with the £48.75 he can buy 24.375 units of good 2 (recall that the price of good 2 is now £2). So the effect of the rise in price of good 2 from £1 to £2 is to increase the gross demand for good 1 from 10 to 16.25 (because the value of the income has risen) and to decrease the gross demand for good 2 from 30 to 24.375 (a combined effect of a rise in income coupled with a fall in the purchasing power of money for good 2).

So with Cobb-Douglas preferences it is easy to compute the (gross) demand for either good: we work out the value of the income; we then allocate a fraction \(a\) to good 1 and a fraction \((1-a)\) to good 2; we then work out how many units of the two goods this allows the individual to buy.

You might like to ask yourself whether this represents your behaviour. Do you allocate fixed fractions of your income to different expenditure categories? Perhaps you do?

Let us now turn to a graphical analysis of a particular case. This ties things up with what we did in chapter 4. Moreover it provides graphical proof that the formula (6.5) above works. I give one example in the text – you can easily provide further examples.

In this example I take the case of an individual with symmetric Cobb-Douglas preferences (that is, \(a = 0.5\)). I assume that he or she has an endowment of 30 units of good 1 and 50 units of good 2 (that
is, $e_1 = 30$ and $e_2 = 50$). The following figure shows his or her indifference curves and the endowment point (point X in the figure).

What he or she should do depends upon the prices of the goods. Let us start with a particular case in which $p_1 = \frac{1}{4}$ and $p_2 = 1$. You should be able to work out that in this case the budget line, as ever, passes through the initial endowment point X, and has slope $= -p_1/p_2 = -\frac{1}{4}/1 = -\frac{1}{4}$. This budget line is shown in the figure.

Let us check that the budget constraint is correct. Clearly it passes through the initial endowment point. At prices $p_1 = \frac{1}{4}$ and $p_2 = 1$ the value of the initial endowment $e_1 = 30$, $e_2 = 50$, is $\frac{1}{4} \times 30 + 1 \times 50 = 57.5$. If the individual wanted to, he or she could spend the whole of this on good 2 – this would enable him or her to reach the point $(0, 57.5)$ – consuming none of good 1 and 57.5 of good 2. This is the point where the budget constraint intersects the vertical axis. In contrast, if the individual wanted to, he or she could spend the whole of the income of 57.5 on good 1 – at a price of good 1 equal to $\frac{1}{4}$ this would enable him to reach the point $(230, 0)$ – consuming 230 units of good 1 and no units of good 2. This point $(230, 0)$ is where the budget constraint intersects the horizontal axis – which you could check by extending the constraint.

Given this budget constraint the best thing that the individual can do is to move to the point that is on the highest possible indifference curve. This is shown in the figure at the point indicated with an asterisk; the highest reachable indifference curve is also indicated. Note an important and perhaps

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1 Later I will change the price of good 1 – which is why the figure has the title that it has.
rather obvious property – the budget constraint is *tangential* to the highest indifference curve at the optimal (asterisked) point.

We know the optimal point. We know that in this symmetric case \( a = 0.5 \) the individual spends half the income on good 1 and half on good 2. The income is 57.5. So the individual spends 28.75 on good 1 and 28.75 on good 2. With 28.75 to spend on good 1 at a price of \( \frac{1}{4} \) the individual can afford 115 units of good 1; with 28.75 to spend on good 2 at a price of 1 the individual can afford 28.75 of good 2. This \((115, 28.75)\) is the asterisked point in the figure. It is the optimal consumption.

To get to \((115, 28.75)\) from the initial endowment point \((30, 50)\) what does the individual do? Sells \(50 - 28.75 = 21.25\) units of good 2, thereby raising 21.25 in money (since the price of good 2 is 1), which he or she spends on good 1, thereby buying 85 more units of good 1 (since the price of good 1 is \(\frac{1}{4}\)), thereby increasing the holdings of good 1 from 30 to 115. The net demand for good 1 is 85 and the net supply of good 2 is 21.25. We have the first column in the following table.

<table>
<thead>
<tr>
<th>Price of good 1</th>
<th>(\frac{1}{4})</th>
<th>(\frac{1}{3})</th>
<th>(\frac{1}{2})</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p_1/p_2) (relative price)</td>
<td>1/4</td>
<td>1/3</td>
<td>(\frac{1}{2})</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>Value of endowment</td>
<td>57.5</td>
<td>60</td>
<td>65</td>
<td>80</td>
<td>110</td>
<td>140</td>
<td>170</td>
</tr>
<tr>
<td>Amount spent on good 1</td>
<td>28.75</td>
<td>30</td>
<td>32.5</td>
<td>40</td>
<td>55</td>
<td>70</td>
<td>85</td>
</tr>
<tr>
<td>Amount spent on good 2</td>
<td>28.75</td>
<td>30</td>
<td>32.5</td>
<td>40</td>
<td>55</td>
<td>70</td>
<td>85</td>
</tr>
<tr>
<td>Gross demand for good 1</td>
<td>115</td>
<td>90</td>
<td>65</td>
<td>40</td>
<td>27.5</td>
<td>23.333</td>
<td>21.25</td>
</tr>
<tr>
<td>Gross demand for good 2</td>
<td>28.75</td>
<td>30</td>
<td>32.5</td>
<td>40</td>
<td>55</td>
<td>70</td>
<td>85</td>
</tr>
<tr>
<td>Net demand for good 1</td>
<td>-21.25</td>
<td>-20</td>
<td>-17.5</td>
<td>-10</td>
<td>5</td>
<td>20</td>
<td>35</td>
</tr>
<tr>
<td>Net demand for good 2</td>
<td>-21.25</td>
<td>-20</td>
<td>-17.5</td>
<td>-10</td>
<td>5</td>
<td>20</td>
<td>35</td>
</tr>
</tbody>
</table>

*If negative indicates a net supply. Note in this table \(p_2 = 1\), \(e_1 = 30\) and \(e_2 = 50\).*

So we have found what the individual should do for a particular set of prices and endowments. For other prices and endowments we simply repeat the procedure. We can therefore discover what happens to the demands and supplies as some variable changes. There are several variables that could vary – specifically the prices of the two goods and the individual’s endowment of the two goods.

Let us begin by exploring how the demands and supplies change when the price of good 1 changes. During this exercise we keep the price of good 2 fixed and the endowments fixed. Specifically \(p_2 = 1\) throughout and \(e_1 = 30\) and \(e_2 = 50\). Let us take 6 more values of \(p_1\); specifically \(1/3, 1/2, 1, 2, 3\) and \(4\). You could check all these out yourself. I will just check one – that with \(p_1 = 1/3\).

If \(p_1 = 1/3\) then the value of the endowment is \(1/3 \times 30 + 1 \times 50 = 60\). If this were all spent on good 2 it would buy 60 units of good 2; if it were all spent on good 1 it would produce 180 units of good 1. So the budget line goes from \((0, 180)\) to \((60, 0)\) and passes through \((30, 50)\) on the way. You could insert this in pencil on the figure above.

Where is the optimal point? The value of the endowment is 60. Our symmetric individual spends half of this, that is 30, on good 1 – thus demanding 90 units of good 1 – and spends the other half on good 2 – thus demanding 30 units of good 2. With initial holdings of 30 and 50 respectively this leads to net demands of 60 and –20 respectively.

You could verify the other entries in the table.

We have found how the demand and supplies for the two goods depend upon the price of good 1. We can take the figures from the table above – specifically the first row and the final two rows and
plot them in a separate figure. This I do here, where I put the price of good 1 on the horizontal axis and the net demands and supplies on the vertical axis.

6.2: changes in the price of good 1

The net demand for good 1 is the curve that starts out positive and then becomes negative. The net demand for good 2 is the straight line that starts negative and then becomes positive. You will notice that when one is positive the other is negative. So if the price of good 1 is sufficiently low (between 0 and 5/3) the individual is a net buyer of good 1 and a net seller of good 2. When the price of good 1 is sufficiently high the individual becomes a net seller of good 1 and a net buyer of good 2. You might ask what is the significance of the price of 5/3? Well, it is the price at which the individual neither wants to be a buyer nor a seller – just wants to stay at the initial position. This is optimal if and only if the budget constraint is everywhere (except at the initial point, of course) below the original indifference curve. That is true if the budget constraint is tangential to the original indifference curve at the initial point. So it follows that the slope of the original indifference curve at the initial point is $-5/3^2$.

The figure shows two things which are generally – but not always – true: (1) that the demand for a good falls as its price rises and (2) that the demand for a good rises as the price of the other good rises. Demand is switched away from the increasingly relatively expensive good towards the other good.

We know the general form of the gross demands. These are given in equation (6.5) above. If we substitute in these the particular values relevant to this example, $e_1 = 30$, $e_2 = 50$ and $p_2 = 1$, we can derive the net demand functions relevant to this case. For good 1 we have that the net demand is given by the gross demand $q_1 = a(p_1 e_1 + p_2 e_2)/p_1$ minus the initial endowment $e_1$. So we have that the net demand for good 1 is given by $0.5(p_1 30 + 50)/p_1 - 30$. Hence the net demand for good 1 is given by $25/p_1 - 15$. This is the equation plotted in the figure – for low $p_1$ it starts out positive, it

2 There is a general property of Cobb-Douglas indifference curves that is useful to know. The marginal rate of substitution (the negative of the slope of the indifference curve) is $aq_2 / [(1-a)q_1]$. For a proof, see the Mathematical Appendix.
becomes zero when \( p_1 = \frac{25}{15} = \frac{5}{3} \) and then it becomes and stays negative. The net demand for good 2 is given by the gross demand \( q_2 = (1-a)(p_1 e_1 + p_2 e_2)/p_2 \) minus the initial endowment \( e_2 \). So we have that the net demand for good 2 is given by \( 0.5(p_1 30 + 50) - 50 \). Hence the net demand for good 2 is given by \( 15 p_1 - 25 \). Note that this is linear in \( p_1 \), as is obvious from the figure. It starts out negative for low \( p_1 \), becomes zero when \( p_1 = \frac{25}{15} = \frac{5}{3} \) and then it becomes and stays positive.

We have now completed our first comparative static exercise – we have studied how the net demands depend upon the price of good 1. Other comparative static exercises are carried out in the same way. These other exercises include changes in the price of good 2, and changes in the endowments of the two goods.

The effect of changes in \( p_2 \) should be easy to predict. Why? Because what is clearly crucial is the slope of the budget constraint – which we know is \(-p_1/p_2\). It is not the absolute value of either price that matters but the relative price. Now above we put \( p_2 \) always equal to 1 and we let \( p_1 \) take the values \( 1/4, 1/3, 1/2, 1, 2, 3, \) and \( 4 \) – so the values of the relative price \(-p_1/p_2\) took the values \( 1/4, 1/3, 1/2, 1, 2, 3, \) and \( 4 \). Now suppose instead that we want to keep the value of \( p_1 \) fixed at 1 and we want to let \( p_2 \) take the values \( 1/4, 1/3, 1/2, 1, 2, 3, \) and \( 4 \). For these values the values taken by the relative price \(-p_1/p_2\) are simply \( 4, 3, 2, 1, 1/2, 1/3 \) and \( 1/4 \). So we can easily find the comparative static effects of changes in \( p_2 \) (for given \( p_1 \), \( e_1 \) and \( e_2 \)). From the table above we get the table below.

<table>
<thead>
<tr>
<th>Price of good 2</th>
<th>1/4</th>
<th>1/3</th>
<th>1/2</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p_1/p_2 ) (relative price)</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>1/2</td>
<td>1/3</td>
<td>1/4</td>
</tr>
<tr>
<td>Value of endowment</td>
<td>42.5</td>
<td>46%</td>
<td>55</td>
<td>80</td>
<td>130</td>
<td>180</td>
<td>230</td>
</tr>
<tr>
<td>Amount spent on good 1</td>
<td>21.25</td>
<td>23%</td>
<td>27.5</td>
<td>40</td>
<td>65</td>
<td>90</td>
<td>115</td>
</tr>
<tr>
<td>Amount spent on good 2</td>
<td>21.25</td>
<td>23%</td>
<td>27.5</td>
<td>40</td>
<td>65</td>
<td>90</td>
<td>115</td>
</tr>
<tr>
<td>Gross demand for good 1</td>
<td>85</td>
<td>70</td>
<td>55</td>
<td>40</td>
<td>65</td>
<td>90</td>
<td>115</td>
</tr>
<tr>
<td>Gross demand for good 2</td>
<td>21.25</td>
<td>23%</td>
<td>27.5</td>
<td>40</td>
<td>65</td>
<td>90</td>
<td>115</td>
</tr>
<tr>
<td>Net demand for good 1*</td>
<td>-8.75</td>
<td>-6%</td>
<td>-2.5</td>
<td>10</td>
<td>35</td>
<td>60</td>
<td>85</td>
</tr>
<tr>
<td>Net demand for good 2*</td>
<td>35</td>
<td>20</td>
<td>5</td>
<td>-10</td>
<td>-17.5</td>
<td>-20</td>
<td>-21.25</td>
</tr>
</tbody>
</table>

*If negative indicates a net supply. Note in this table \( p_1 = 1 \), \( e_1 = 30 \) and \( e_2 = 50 \).

Note the implications – in particular the final two rows of this table are the same as the final two rows of the first table – and simply because we have exactly the same relative prices - except in the reverse order. If we plot the final two rows of this table against the first row we get the following figure:

5.4: changes in the price of good 2
Note that the variable on the horizontal axis is the price of good 2. Note also that the curve in the above figure is the net demand for good 2: it starts out positive for low \( p_2 \), becomes negative when \( p_2 = 3/5 \) (what is the significance of this?) and then becomes and stays negative. The line is the net demand for good 1: it starts out negative, becomes zero at a price of 3/5 and then becomes and stays positive.

The final comparative static exercise that we perform with these preferences is that of varying one of the endowments. Specifically we vary \( e_1 \). At the same time we keep prices constant – and for simplicity we put the prices equal so that the budget line has a slope \(-1\). We also keep the endowment of good 2 fixed at 50. We vary \( e_1 \) starting with a value of 10. This is pictured in the next figure.

You can probably see where the optimal point is. As everything is symmetrical the optimal point is too – it is indicated with an asterisk and is at (30, 30). The net demands are 20 for good 1 and \(-20\) for good 2. What do you think happens when we increase \( e_1 \) to 20? The optimal point is at (35, 35) and the net demands are 15 for good 1 (slightly less than before because the initial endowment of good 1 has increased) and \(-15\) for good 2. If we continue to increase \( e_1 \) in this way, the gross demand for good 1 continues to increase (because the individual is getting better off) but the net demand continues to fall. By the time that \( e_1 \) has reached 50 (the same as \( e_2 \)) the net demand for good 1 has sunk to zero. Thereafter it becomes negative. The following graph illustrated the situation:

You have now done 3 comparative static exercises. You should be able to do a fourth (changes in the endowment of good 2 with fixed prices and fixed endowment of good 1) yourself.
We do not want to get involved in too much detail. What is important is not that you understand the detail but rather understand the principles involved in finding the optimal position and hence in seeing how the individual responds to changes in the exogenous variables. Also of importance is that you have some feeling how the shape of the various ‘demand functions’ (figures 6.2, 6.4 and 6.6) are influenced by the underlying preferences – as manifested in the indifference curves. To reinforce the feeling let me take some slightly different preferences – non-symmetrical Cobb-Douglas – and see how one of the figures differs. Take $a = 0.3$ so that $(1-a) = 0.7$. Either from the indifference map below

or from the formula above we can derive the following relationship between net demands and the price of good 1.

This should be compared with figure 6.2. You will see differences in (1) the slopes; (2) the point at which the individual switches from being a buyer of one good to being a seller of it. Clearly the preferences determine the demand functions.

6.4: Choice with Stone-Geary Preferences

To reinforce this latter point, let us now turn to Stone-Geary preferences. As before we will find it simplest to begin with a mathematical derivation of the demand functions. Now it may be thought that this will be rather difficult, but it turns out that we can use a simple trick. Recall that with Stone-Geary preferences the individual has subsistence levels of the two goods, but then his or her indifference curves, relative to these subsistence levels, are Cobb-Douglas. So – recalling that a Cobb-Douglas individual spends a constant fraction $a$ of his or her income on good 1 and a constant
fraction \((1-a)\) of his or her income on good 2, we can argue the following: first a Stone-Geary person buys a quantity \(s_1\) of good 1 and a quantity \(s_2\) of good 2, and then he or she spends a constant fraction \(a\) of his or her residual income on good 1 and a constant fraction \((1-a)\) of his or her residual income on good 2. This leads us to the following gross demand functions:

\[
q_1 = s_1 + a(p_1e_1 + p_2e_2 - p_1s_1 - p_2s_2)/p_1 \quad \text{and} \quad q_2 = s_2 + (1-a)(p_1e_1 + p_2e_2 - p_1s_1 - p_2s_2)/p_2 \quad (6.7)
\]

You should compare these with the Cobb-Douglas demands – given in (6.5). Notice that they have different properties. This implies that if we know the demand functions we may be able to infer the underlying preferences.

### 6.5: Choice with Perfect Substitute Preferences

If the individual regards the two goods as perfect substitutes the decision problem is particularly simple: the individual buys only the cheapest of the two goods – where ‘cheapest’ is defined relative to his or her preferences. For 1:1 substitutes it is clear: if \(p_1 < p_2\) then the individual buys only good 1; if \(p_2 < p_1\) then the individual buys only good 2. For 1:a substitutes, we should note that the individual regards 1 unit of good 1 as being the same as \(a\) units of good 2. The former costs \(p_1\) the latter \(ap_2\). So if \(p_1 < ap_2\) then the individual buys only good 1; if \(ap_2 < p_1\) then the individual buys only good 2. We can illustrate the first of these two cases:

**6.18: changes in the price of good 1 with perfect 1:1 substitutes**

Here we have perfect 1:1 substitutes. The price of good 2 is assumed to be 1 and the endowments are 30 and 50. The price of good 1 in this figure is \(1/4\) - the budget line passes through the initial endowment point and has a slope of \(-1/4\). The maximum that the individual can buy of good 2 is 57.5, the maximum that he or she can buy of good 1 is 230. The figure shows the optimal point – when the individual spends all his income on good 1. His gross demand for good 1 is 230 and for good 2 is 0. Therefore the net demands are 200 and \(-50\): the individual sells all his or her 50 units of good 2 and with the proceeds buys 200 more units of good 1.

Clearly the same kind of solution is valid whenever \(p_1 < p_2\). When they are equal the budget constraint coincides with an indifference curve – so any point along the budget constraint is optimal. When \(p_2 < p_1\) the situation changes and the individual sells all his endowment of good 1 and with the proceeds buys good 2. We get the following table:

<table>
<thead>
<tr>
<th>Price of good 1</th>
<th>1/4</th>
<th>1/3</th>
<th>1/2</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gross demand for good 1</td>
<td>230</td>
<td>180</td>
<td>130</td>
<td>*</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Gross demand for good 2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>*</td>
<td>110</td>
<td>140</td>
<td>170</td>
</tr>
<tr>
<td>Net demand for good 1</td>
<td>200</td>
<td>150</td>
<td>100</td>
<td>*</td>
<td>-30</td>
<td>-30</td>
<td>-30</td>
</tr>
<tr>
<td>Net demand for good 2</td>
<td>-50</td>
<td>-50</td>
<td>-50</td>
<td>*</td>
<td>60</td>
<td>90</td>
<td>120</td>
</tr>
</tbody>
</table>
*Any amount satisfying the budget constraint. Note in this table $p_2 = 1$, $e_1 = 30$ and $e_2 = 50$.

Notice how the gross demand switches from good 1 to good 2 as the price of good 1 rises through 1. If we graph the final two rows of this table against the first row, we get the following net demand functions for the two goods as functions of the price of good 1.

![Graph showing net demands for two goods](image)

This may look a bit odd but you should read it carefully. It says that at any price between 0 and 1 the individual sells all 50 units of good 2 and with the proceeds buys as much good 1 as possible; for prices above 1 the individual sells all 30 units of good 1 and with the proceeds buys as much good 2 as possible. Mathematically the gross demands are:

$$
\begin{align*}
\text{if } p_1 < p_2 & \text{ then } q_1 = \frac{(p_1 e_1 + p_2 e_2)}{p_1} \text{ and } q_2 = 0 \\
\text{if } p_2 < p_1 & \text{ then } q_2 = \frac{(p_1 e_1 + p_2 e_2)}{p_2} \text{ and } q_1 = 0 
\end{align*}
$$

(6.8)

The net demands are found by subtracting the initial holdings, and are thus as follows:

$$
\begin{align*}
\text{if } p_1 < p_2 & \text{ then for good 1 } (p_1 e_1 + p_2 e_2)/p_1 - e_1 \text{ and for good 2 } -e_2 \\
\text{if } p_2 < p_1 & \text{ then for good 2 } (p_1 e_1 + p_2 e_2)/p_2 - e_2 \text{ and for good 1 } -e_1 
\end{align*}
$$

(6.9)

For perfect 1:a substitutes we have the following obvious generalisation for the gross demands:

$$
\begin{align*}
\text{if } p_1 < ap_2 & \text{ then } q_1 = \frac{(p_1 e_1 + p_2 e_2)}{p_1} \text{ and } q_2 = 0 \\
\text{if } ap_2 < p_1 & \text{ then } q_2 = \frac{(p_1 e_1 + p_2 e_2)}{p_2} \text{ and } q_1 = 0 
\end{align*}
$$

(6.10)

The derivation of the net demands is left up to you.

One thing should be very obvious – there is a startling difference between the demand functions with Cobb-Douglas preferences and those with perfect substitute preferences – just compare figure 6.2 with figure 6.19!

### 6.6: Choice with Perfect Complement Preferences

This is a relatively easy case as the individual has to buy the two goods in fixed proportions. Why? Just examine the optimisation problem, here with 1-with-1 perfect complements.
Where is the optimal point? Always at a corner. So the optimal solution is simply where the budget constraint and the line joining the corners intersect each other. In the case of 1-with-1 perfect complements the line joining the corners has equation $q_1 = q_2$ whereas the budget constraint has equation $p_1 q_1 + p_2 q_2 = p_1 e_1 + p_2 e_2$. Solving these two equations simultaneously we get the following gross demands:

$$q_1 = q_2 = \frac{(p_1 e_1 + p_2 e_2)}{(p_1 + p_2)}$$  \hspace{1cm} (6.11)

The net demands can easily be found – by subtracting $e_1$ from $q_1$ and $e_2$ from $q_2$.

We illustrate the implications for the net demands in the case above, where $e_1 = 30$, $e_2 = 50$, $p_2 = 1$ and where we plot the net demands as functions of the price of good 1.

Notice that the individual is always a net demander of good 1 and a net supplier of good 2. Why?

For perfect 1-with-a complements the line joining the angles has equation $q_2 = a q_1$ and so the optimal gross demands are the intersection of this line and the budget constraint. This gives us the following gross demands:

$$q_1 = \frac{(p_1 e_1 + p_2 e_2)}{(p_1 + ap_2)} \quad \text{and} \quad q_2 = a(p_1 e_1 + p_2 e_2)/(p_1 + ap_2)$$  \hspace{1cm} (6.12)

It is left up to you to find the net demands.

\textbf{6.7: Comments}
The main point has already been made – the demands depend upon the preferences. If we compare figures 6.2, 6.10, 6.19 and 6.23, this point is very clear. This point is important not least because it enables us to infer preferences from observations. Why might we want to do this? If we have some observations on an individual’s demand and want to make some predictions about future demand. First we use the observations on demand to tell us something about preferences and then we use that information to predict future demand. This is the economists’ (or any scientists’) methodology.

6.8: Summary

You may have already realised that we can distinguish between some of the preferences we have studied here quite easily. We see:

With perfect 1:a substitutes the individual buys only one of the goods switching from one to the other at a price of a.

With perfect 1-with-a complements the ratio of the gross demand for good 2 to the gross demand for good 1 is always constant and equal to a.

With Cobb-Douglas preferences with parameter a the ratio of the expenditure on good 1 to the expenditure on good 2 is always constant and equal to a/(1-a).

You might like to see if you can discover a similar kind of rule for Stone-Geary preferences.

6.9: Mathematical Appendix

We derive the Cobb-Douglas demands. It will be recalled from section 6.3 that the problem is to find the solution to the problem of the maximisation of \( U(q_1,q_2) = a \ln(q_1) + (1-a) \ln(q_2) \) subject to the budget constraint \( p_1q_1 + p_2q_2 = p_1e_1 + p_2e_2 \).

Possibly the easiest way is to set up the Lagrangian (the objective minus \( \lambda \) times the constraint) :

\[
L = a \ln(q_1) + (1-a) \ln(q_2) - \lambda (p_1q_1 + p_2q_2 - p_1e_1 - p_2e_2)
\]

and maximise \( L \) with respect to \( q_1, q_2 \) and \( \lambda \).

The optimality conditions are:

\[
\frac{dL}{dq_1} = \frac{a}{q_1} - \lambda p_1 = 0
\]
\[
\frac{dL}{dq_2} = \frac{(1-a)}{q_2} - \lambda p_2 = 0
\]
\[
\frac{dL}{d\lambda} = p_1q_1 + p_2q_2 - p_1e_1 - p_2e_2 = 0
\]

From the first of these we have \( p_1q_1 = a/\lambda \), from the second \( p_2q_2 = (1-a)/\lambda \) and so, using the third, \( 1/\lambda = p_1e_1 + p_2e_2 \). From these we get:

\[
q_1 = \frac{a(p_1e_1 + p_2e_2)}{p_1} \quad \text{and} \quad q_2 = \frac{(1-a)(p_1e_1 + p_2e_2)}{p_2}
\]

which are equation (6.1) of the text.
An alternative proof uses the fact that at the optimal point, the slope of the indifference curve is equal to the slope of the budget line. We know the latter – it is just \(-\frac{p_1}{p_2}\) - and the former is minus the *marginal rate of substitution*. We can work this out from the equation of an indifference curve, which is given by \(a \ln(q_1) + (1-a) \ln(q_2) = \text{constant}\). If we differentiate this equation we get \(a \frac{dq_1}{q_1} + (1-a)\frac{dq_2}{q_2} = 0\) from which it follows that the slope of the indifference curve \(\frac{dq_2}{dq_1} = -\frac{aq_2}{(1-a)q_1}\). Imposing the condition that at the optimal point the slopes of the budget line and the indifference curve are equal, we get \(-\frac{p_1}{p_2} = -\frac{aq_2}{(1-a)q_1}\) from which it follows that \(\frac{p_1}{p_2} = \frac{aq_2}{(1-a)q_1}\). Combining this with the budget constraint gives us the Cobb-Douglas demands above.