Chapter 3: Discrete Goods: Reservation Prices, Demand and Supply, and Surpluses

3.1: Introduction

We continue to work with a discrete good - that is, one that can be bought and sold only in integer units. In this chapter we explain where the reservation prices that we assumed in chapter 2 ‘come from’, and we generalise the analysis of chapter 2 so that the individual may buy or sell more than one unit of the good and may, depending upon the price of the good, be either a buyer or a seller of the good. We explain the circumstances under which he or she is a buyer and under which he or she is a seller. Finally, we show more formally the results that the buyer surplus is the area between the price paid and the demand curve, and that the seller surplus is the area between the price received and the supply curve. These latter results are of particular importance in practice – where we might have an estimate of the supply and demand curves and want to calculate the effects of some proposed policy measure.

3.2: The Initial Position

We are interested in a particular individual’s demand and supply for some particular good. I am going to use a graphical analysis and I put the quantity of the good on the horizontal axis. In addition to the particular good in which we are interested, we suppose that the individual has money to spend on other goods and I put the quantity of money that the individual has on the vertical axis. Take the units of this to be £. Let us assume that the individual starts with an initial endowment of
some units of the good (this could be zero) and some amount of money (this could be zero). In the example that follows it is assumed that the individual starts with 3 units of the good and 30 units of money – that is £30. This initial endowment point is indicated in the figure 3.1 with the letter ‘X’.

The individual starts at point X. Perhaps through trade he or she can move to some other point in the space. Let us see what we can say about other points as compared with the point X. You will see that in the graph the space has been divided up into four quadrants based on the point X. It should be clear that any point in the north-east quadrant must be preferred by the individual to the point X – assuming that he or she likes the good and likes money – as all points in this north-east quadrant have more of at least one. Similarly we can say that he or she prefers point X to all the points in the south-west quadrant – as point X gives him or her more of at least one of money or the good than any point in the quadrant. What about the remaining two quadrants? In each of these, he or she has either more money and less of the good, or less money and more of the good, than point X. So, unless we know more about the individual’s preferences, we cannot say whether he prefers point X or some point in the quadrant.

3.3: Indifference
Let us suppose that we know something about the preferences of the individual as far as this good is concerned. In particular, let us suppose that we know that the individual would pay at most £5 to buy one extra unit of the good – that is, we know that his reservation price for the first extra unit of the good is £5. What does this tell us? The individual starts at the point (3, 30) – that is with 3 units of the good and £30. If he or she were to spend £5 acquiring one extra unit of the good he or she would move to the point (4, 25) – with one extra unit of the good and £5 less in money. If this £5 is the maximum that he or she would pay then we can conclude that the individual is indifferent between (3, 30) and (4, 25). Graphically he or she is indifferent between point X and the point labelled ‘4’ in the figure 3.2.

We should be clear that we understand what is meant by this: indifference is an extremely important property to economists but its meaning is not always obvious to non-economists. When we say that the individual is indifferent between point X and point 4 we mean that the individual does not care whether he or she is at point X (has 3 units of the good and 30 units of money) or is at point 4 (has 4 units of the good and 25 units of money). In fact, because of this indifference the individual would be perfectly happy if someone else (or some machine) decided whether he was to be at point X or at point 4. Furthermore, an implication of this indifference is that if the individual was offered a choice between point X and some point better than point 4 (for example, the point (4, 26) – check that you are clear that this is better than point 4) then he or she would choose the point better than
point 4. Furthermore, if offered a choice between point X and a point worse than point 4 (for example, the point (4, 24)) the individual would choose point X.

So far we have assumed that we know that the individual’s reservation value for the first extra unit of the good is £5 – that is, £5 is the maximum amount of money that he or she would pay to have 4 instead of 3 units of the good. Obviously this ‘5’ depends upon the preferences of the individual and how he or she feels about the good vis-à-vis money.

Let us suppose we know some more. Suppose we know that, after acquiring this first extra unit, the individual would pay at most £3 to acquire a second extra unit and would then pay at most £2 to acquire a third extra unit of the good. This tells us that the individual’s reservation price for the second extra unit is £3 and the reservation price for the third extra unit is £2. This information also tells us that the individual is indifferent between (3, 30) and (4, 25) (because his or her reservation price for the first unit is £5) and (5, 22) (because his or her reservation price for the second unit is £3) and (6, 20) (because his or her reservation price for the third unit is £2).

We might also have information about the individual’s preferences in the other direction. Recall that the individual starts at (3, 30) – with 3 units of the good and 30 units in money. Suppose we are told that the individual would sell one of his three units of the good if he was given sufficient extra money. Suppose also that we are told that 10 units of money are the least that he or she would accept to sell one of the three units of the good. Then this information tells us that the individual is indifferent between the initial point (3, 30) and the point (2, 40) – where he or she has one less unit of the good but £10 more in money. Finally let us suppose that we know that the individual would sell a second unit of the good – but only for a minimum compensation of 30 units of money. This tells us that the individual is indifferent between point X (3, 30) and the point (2, 40) and the point (1, 70).
Let us show all of this graphically. We have that the individual is indifferent between point X and the points 1, 2, 4, 5 and 6 in figure 3.3.

If we join together these points we get what is called an ‘indifference curve’ – the individual is indifferent between all the points along it.

Moreover, it follows from what has gone before that the individual prefers any point above and to the right of this indifference curve to any point on it, and prefers any point on it to any point below and to the left of it.

\(^1\) Though note, of course, that in this discrete good case, the points in between integer values of the good have no real meaning.
3.4: Reservation Prices

To construct this indifference curve, we started with knowledge of the individual’s reservation prices. We can, of course, work in the other direction: given the indifference curve we can work out the individual’s reservation prices. Given the initial endowment of (3, 30) the individual would be a buyer for sufficiently low prices – specifically, he or she has the following reservation prices: for the first unit, £5; for the second unit, £3; for the third unit £2.

For sufficiently high prices he or she would be a seller – specifically, he or she has the following reservation prices: for the first unit, £10; for the second unit, £30.

Note that these obviously depend upon his or her initial endowment. If instead of being endowed with 3 units of the good and £30 I money, he or she was endowed with 4 units and £25 in money (notice that this point is a point on the indifference curve that we have drawn), then his or her reservation prices would be the following. First, as a buyer: for the first unit, £3; for the second unit, £2. Second, as a seller: for the first unit, £5; for the second unit, £10; for the third unit, £30.

3.5: Indifference Curves

In the above analysis we assumed that the initial endowment point of the individual was as (3, 30) – that is with 3 units of the good and £30 in money. Obviously we can repeat the analysis for any initial endowment point – and we can therefore derive an indifference curve through any point in the space. The shape of these indifference curves will depend upon the individual’s preferences – though they will always be downward-sloping (because we have assumed that the individual likes both the good and money). However if we make a particular assumption about these preferences
then we can derive any other indifference curve from the one that we have already derived. As will be realised, this is a strong assumption, and in future chapters we will weaken it, but it will prove useful to us throughout this chapter and the next. This assumption is the following:

2 The individual’s reservation prices for the good depend upon the number of units of the good but do not depend upon the amount of money that the individual has.

What does this imply? Consider the individual’s indifference curve through the initial endowment point (3, 40) – which differs from that above in that the initial endowment of income is £10 higher. If we assume that the reservation prices are the same as when the initial endowment point is (3, 30) then we can draw the following indifference curve through the new initial point:

![Figure 3.5: a second indifference curve](image)

Note that, for example, moving from point X to point 4 in this figure involves giving up £5 in exchange for an extra unit of the good – just as before. The reservation prices implied by the indifference curve in figure 3.5 are exactly the same as those implied by the indifference curve in figure 3.4.

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2 For reasons which will become clear later, this is referred to as the assumption that the individual’s preferences are quasi-linear.
If we compare figures 3.4 and 3.5 what do we see? That the indifference curve in figure 3.5 is parallel in a vertical direction to that in figure 3.4. Or, in other words, the vertical distance between the two curves is everywhere £10.

Let us assume that we the indifference curves always have this property. Then, if we draw indifference curves through the points (3, 30), (3, 40), (3, 50), (3, 60), (3, 70) and (3, 80) we have:

Note – each curve is a constant distance away from any other – in a vertical direction. The indifference curves are parallel in a vertical direction. It follows that we can conclude a number of things: (1) the higher the indifference curve on which the individual is the happier he or she is; (2) we can measure how much happier he or she is by the vertical distance between the relevant two indifference curves. For example, if he or she is on the top indifference curve rather than the bottom one (in the figure above) then the individual is £50 better off – because the top indifference curve is everywhere £50 above the bottom indifference curve\(^3\).

Obviously, maintaining the assumption above (that the reservation prices are independent of the initial endowment of money), all the above can be generalised: through every point in the space above there is an indifference curve – and all the indifference curves are parallel in a vertical direction.

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\(^3\) Check that you are happy with this. One point on the top indifference curve is (3,80) and one point on the bottom indifference curve is (3,30). How much better off is the individual at (3,80) as compared with (3,30)? That is, how much
direction – implying that this vertical distance is a measure of the change in well-being of the individual in moving from one indifference curve to another.

### 3.6: Implied Demand Curve

What is the individual’s implied demand curve? We have seen that the individual has a reservation price of £5 for the first unit purchased, a reservation price of £3 for the second unit purchased and a reservation price of £2 for the third unit purchased. It follows that, starting from point X, the individual’s demand function for the good is a step-function with steps at the prices £5, £3 and £2. The demand curve is shown in the figure 3.10, which has the quantity of the good on the horizontal axis and the *price of the good* on the vertical axis. (Note that this is a different space from that used in the figures above.)

![3.10: implied demand curve](image)

We check that this has the correct properties. It says that at any price greater than £5 the individual would not demand any units of the good. At a price between £5 and £3 (the reservation prices for the first and second units purchased) the demand would be 1 unit. At a price between £3 and £2 (the reservation prices for the second and third units purchased) the demand would be 2 units. At a price...
less than £2 (the reservation price for the third unit) the demand would be 3 units. (I am ignoring the possibility of more than 3 units being purchased.) The demand function is a step-function with a step at every reservation price.

Let us consider some particular examples. Let us also work out the gains from trade in these particular examples and let us show that our general result (that the surplus is equal to the area between the price paid and the demand curve) is true in this context.

Consider a price of £4. If the individual is offered the opportunity to buy or sell the good at a fixed price of £4 per unit, what would he or she do? To answer this, let us consider the opportunities offered by the existence of this price.

The individual starts at point X – with 3 units of the good and £30. Given a price of £4 per unit, we can work out where he or she could move to with different trades. For example, if he or she buys just one unit at a price of £4 per unit then he or she moves from the point (3, 30) to the point (4, 26) – he or she would have 1 more unit of the good but £4 less in money. If two units are bought he or she moves from (3, 30) to (5, 22) – the two units purchased costing £8. If three units are purchased then he or she moves from (3, 30) to (6, 18). And so on. Alternatively he or she could sell. If he or she sells one unit at price of £4 per unit he or she moves from (3, 30) to (2, 34); if two units are sold
at a price of £4 the move is from (3, 30) to (1, 38). And so on. So a price of £4 per unit offers the individual the opportunity to either stay at (3, 30) or move to any one of the points (0, 42), (1, 38), (2, 34), (4, 26), (5, 22) or (6, 18). If we join these points up we get the straight line through the point X in figure 3.11. Note very importantly that this has slope -£4 – equal to (minus) the price of the good, since each unit bought decreases the money holdings by the price, and since each unit sold increases the money holdings by the price. This line is given the name the budget constraint of the individual – it shows the possibilities offered to the individual by a price of £4 per unit.

The budget constraint shows the opportunities open to the individual. What does he or she do? In principal it is easy to answer this – he or she chooses the point on it which is highest in terms of his or her indifference map – that is he or she chooses the point which is highest in a vertical direction from the indifference curve passing through the initial point. From the figure above, it is clear that all the points to the left of point X lie below this initial indifference curve – so the individual would be worse off moving to the left, that is, by selling. The reason is simple – the price is too low. Furthermore we know the point (5, 22) is on the original indifference curve and that the point (6, 18) lies below it. This just leaves the point (4, 26) – which is clearly on a higher indifference curve than originally. In fact we can work out that it must be on an indifference curve £1 higher – as the point (4, 25) lies on the original curve and the point (4, 26) is £1 higher in a vertical direction than (4, 25).

It follows that of all the points (with integer values for the quantity of the good) on the budget constraint for a price of £4 per unit, the best is unambiguously the point (4, 26). Accordingly the individual moves there – by buying one unit at a price of £4 – and is, as a consequence, £1 better off\(^4\). This £1 is the surplus the individual has when offered a price of £4 for the good.

\(^4\) This improvement can also be seen another way. At a price of £4 the individual buys one unit – for which his or her reservation price is £5. He was willing to pay up to £5 for this unit but only paid £4 – and is therefore £1 better off as a consequence.
If we now return to the demand curve, we can show that our previous result about this surplus is still valid. We have:

At a price of £4 per unit the individual demands one unit. Moreover the area between the price paid and the demand curve – the shaded area in the figure above – is exactly equal to £1 – the surplus of the individual when offered a price of £4. Our result that the surplus is equal to the area between the price paid and the demand curve remains true.

This result is both profound and trivial (trivial if you are a mathematician) but it is important. Fully understanding why it is true may take a little time. In the meantime, I am happy if you are satisfied that it works in these numerical examples. To reinforce your understanding let me give two more examples. After you have studied these you could try some examples of your own.

Consider now a lower price – say a price of £2.5 per unit. The figure below shows the budget constraint implied by this price – the straight line through the initial point. Note that it passes through the points (0, 37.5) (the individual sells all three units), (1, 35) (the individual sells 2 units), (2, 32.5) (the individual sells one unit), (3, 30) (the individual neither buys or sells), (4, 27.5) (the individual buys one unit), (5, 25) (the individual buys 2 units) and (6, 22.5) (the individual buys 3 units). Note also that this budget line has a slope of -£2.5.
Once again it is clear that the individual is worse off if he sells any units. At (4, 27.5) he is £2.5 better off than originally (the original indifference curves passes through (4, 25)). At (3, 25) he is £3 better off than originally (the original indifference curves passes through (3, 22)). At (6, 22.5) he is £2.5 better off than originally (the original indifference curves passes through (6, 20)). Clearly he or she is best off at (5, 25) – that it by buying 2 units – and there he or she is better off by £3. His or her surplus from the trade is £3. Let us now return to the demand curve:

3.14: profit/surplus at a price of £2.5

This confirms that his or her demand at a price of £2.5 is 2 units of the good. The surplus is the area between the price of £2.5 and the demand curve – the shaded area above. It is equal to £2.5 plus £0.5 = £3, exactly as we have derived above. The result is once again confirmed.

Finally let us consider a price of £1.8 per unit. The implied budget constraint is the straight line in the figure below – note that it has a slope of - £1.8.
Where is the best point on this budget constraint? You should be able to verify that it is at (6, 24.6) – to which the individual moves by buying 3 units (at a cost of £5.4) – and at which the individual is £4.6 better off than originally. His or her surplus from the trade is £4.6. In terms of the demand function:

At a price of £1.8 per unit the demand is 3 and the surplus is the shaded area – which is equal to £3.2 plus £1.2 plus £0.2 – which is equal to £4.6 – the surplus made by the individual when offered a price of £1.8.

The crucial point about all of the above is that we have verified the result that

*The buyer surplus is equal to the area between the price paid and the demand curve.*
It should be noted that this result does rely on the assumption that we made above – that the indifference curves are parallel in a vertical direction – that is, that the reservation prices are independent of the initial money endowment. This assumption is called the assumption that the individual has *quasi-linear preferences*. We shall discuss this further in the conclusions.

### 3.7: Implied Supply Curve

In the section above we took as examples prices sufficiently low so that the individual chose to be a buyer of the good. However, in general, it is normally the case that if the individual is endowed with some units of the good, there will be prices sufficiently high to induce the individual to be a seller. In fact we know this already – because we know that the individual will sell one of the units of the good at the reservation price of £10 and will sell a second at a reservation price of £30. The implied supply curve is thus:

![Graph of implied supply curve](image)

We check that this has the correct properties. It says that at any price less than £10 the individual would not sell any units of the good. At a price between £10 and £30 (the reservation prices for the first and second units sold) the supply would be 1 unit. At a price greater than £30 (the reservation
price for the second unit) the supply would be 2 units. (I am ignoring the possibility of more than 2 units being sold.) The supply function is a step-function with a step at every reservation price.

As in the case of demand let us consider some particular examples. Consider first a price of £20 per unit. The budget constraint is shown in the figure below – the straight line passing through the initial point X. It has a slope equal to -£20 – that is, equal to (minus) the price of the good.

Where is the best point on this budget constraint? Obviously the individual does not want to buy the good – does not want to move the right of the initial point – as he or she is worse off there in the sense that he or she is on a lower indifference curve than originally. Nor does he or she want to sell 2 or more units. Indeed it is clear that the best point (restricting ourselves as ever to integer values of the good) is (2, 50) – to which the individual moves by selling one unit of the good. At (2, 50) he or she is better of by £10 than originally – is on an indifference curve £10 vertically higher than originally (note that (2, 40) is on the original indifference curve).
Let us check this with the supply curve:

At a price of £20 the supply is one and the surplus is the shaded area in the figure above – the area between the price received and the supply curve. It can easily be seen that this area is £10 – exactly equal to the surplus gained from trading at the price of £20\(^5\).

Let us present one final example. Consider a price of £32. The budget constraint is shown in the figure below – it passes through the initial endowment point (the individual can always choose to remain there) and has a slope of -£32 – equal, as ever, to minus the price of the good.

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\(^5\) Another way of checking the surplus calculation is to note that the individual’s reservation price for the first unit sold is £10 – and hence there is a surplus of £10 if the individual sells that first unit for £20.
Given the budget constraint (and the restriction to integer values for the good) the individual can move to (0, 126), (1, 94), or (2, 62) or stay at (3, 30). (Note that the individual does not have enough money to buy even one unit of the good.) The point (0, 126) is on a lower indifference curve than the original one. The point (1, 94) is on an indifference curve £24 higher than the original one (note that (1, 70) is on the original curve). The point (2, 62) is on an indifference curve £22 higher than the original (note that (2, 40) is on the original). So the best thing for the individual to do is to move from (3, 30) to (1, 94) by selling two units of the good – therefore ending up £24 ‘better off’ than originally – through being on an indifference curve £24 higher than the original one. In terms of the supply curve, we have:

We see that at a price of £32 the supply is 2 units and the surplus gained from the trade is the shaded area in the figure above – which is equal to £22 + £2 = £24, exactly the same as we have already demonstrated.
The crucial point about all of the above is that we have verified the result that

*The seller surplus is equal to the area between the price received and the supply curve.*

Once again it should be noted that this result does rely on the assumption that we made above –that the preferences of the individual are *quasi-linear* – that is, that the indifference curves are parallel in a vertical direction. We shall discuss this further in the conclusions.

### 3.8: Comments

First I should comment about the fact that this whole chapter has used a specific example. You might be wondering why I have done this and how much of the detail you need to understand and remember. The reason that I have used a specific example throughout is so that the various results that I have been claiming can easily be confirmed – the calculations involved are simple arithmetic operations. So one reason for using a specific example is to build up your confidence in the various results that I am claiming (some of which can only be proved formally with rather complex mathematics). However, I do not want you to get bogged down with detail and I certainly do not want you to memorise any details of the special example that I have used. What I want you to concentrate attention on are the methods I have used, the principles I have used. I would hope that you get out of this chapter the following:

1) That quasi-linear preferences (parallel-in-a-vertical-direction indifference curves) are important because they enable us to measure how much better or worse off an individual is if he or she moves from one point in the space to another; this, in turn, enables us to measure the surplus gained when buying or selling.
2) That the budget constraint in \((q, m)\) space must have slope equal to minus the price of the good.

3) That the best place for the individual to be given a particular budget constraint is to be as high as possible vertically relative to the original indifference curve – given the restriction to integer values for the quantities.

4) That by using this procedure we can find the best strategy for the individual at any price – we can compute the gross demand for the good at any price.

5) If this gross demand is greater than the initial endowment then the individual wants to buy more of the good – his or her net demand is positive. By plotting these net demands against the price we get the demand curve, and if we look at the area between the price paid and the demand curve we find the surplus made by the individual as a result of the purchase – that is, we find out how much better off he or she is as a consequence of the purchase.

6) If this gross demand is less than the initial endowment then the individual wants to sell some of the good – his or her net supply is positive. By plotting these net supplies against the price we get the supply curve, and if we look at the area between the price received and the supply curve we find the surplus made by the individual as a result of the sale – that is, we find out how much better off he or she is as a consequence of the sale. These results concerning the surpluses should be fairly obvious in this context (the integer-only context).
We have shown that the two key results concerning the buyer and seller surplus are true in the context of this chapter. This chapter is special in two key respects: first, the good is discrete; second, I have assumed that the preferences are such that the indifference curves are parallel in a vertical direction. The two key results do not depend on the discrete nature of the good – as I will show in the next chapter – but they do depend crucially on the assumption about the indifference curves. As has already been noted, this parallel property is known in the literature as the assumption that the preferences are quasi-linear. The important intuitive point is that it means that reservation prices for the good do not depend upon the amount of money that the individual has. Is this true for you?

Suppose you have no units of some good. Ask yourself whether how much you are willing to pay to have one unit of the good depends upon how much money you have. If it does you do not have quasi-linear preferences; if it does not you may have.

You should understand why this assumption is important. It implies that indifference curves are parallel in a vertical direction, so that if at some quantity of the good one curve is £10 higher than another then it is £10 higher at any quantity of the good. This means that we can unambiguously say that the individual is £10 better off if on the higher curve than on the lower curve – we have an unambiguous measure of how much better off is the individual. If indifference curves are not parallel then we do not have such an unambiguous measure. We will explore the implications in later chapters.

I have spent some considerable time deriving and discussing the key results about the measurement of the surplus. Perhaps you are asking why? These key results are important for a number of reasons, not least when we want to assess the impact of some policy change. Often such a policy change will change the price in some market and we may want to calculate the effect on the welfare

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6 The reason for this is that the equation of an indifference curve in \((q,m)\) space (where \(q\) denotes the quantity of the good and \(m\) denotes the quantity of money) is given by an equation of the form: \(m + f(q) = \text{constant}\), where \(f(.)\) is some decreasing function. That is the indifference curves are partly linear (in \(m\)) and partly non-linear.
of the participants in the market. If the price goes up it makes the buyers worse off and the sellers better of. We can calculate how much better and worse off using the key results. The loss in the buyer surplus is the area between the demand curve and the old and new prices. Similarly the increase in the seller surplus is the area between the supply curve and the old and new prices. Often we have estimates of the demand and supply curves (see chapter 16) and hence we can estimate the losses and gains in the surpluses. This will help us to decide whether to implement the policy change or not.

3.9: Summary

We have restricted attention in this chapter to the case of a discrete good – one that can be bought and sold only in integer units. Moreover, for most of the chapter we have assumed that the preferences take the particular form of quasi-linear preferences. However, independent of these two restrictions we have defined the key concept of an indifference curve:

An indifference curve is a locus of points about which the individual feels indifferent.

We have defined what we mean by a budget constraint and shown that it has the important property that:

The slope of the budget constraint in (quantity of the good, money) space is simply equal to minus the price of the good.
Moreover we have shown how to derive demand and supply curves from indifference curves. In particular we have shown that:

The demand function takes the form of a step-function with a step at every reservation price.

The supply function takes the form of a step-function with a step at every reservation price.

The individual is a buyer for sufficiently low prices and a seller for sufficiently high prices.

Moreover we have verified two key results from chapter 2:

The buyer surplus is the area between the price paid and the demand curve.

The seller surplus is the area between the price received and the supply curve.

Finally we recall the definition of quasi-linear preferences:

*With quasi-linear preferences the indifference curves are parallel in a vertical direction and the reservation prices are independent of the amount of money held by the individual.*

### 3.10: Review Questions

1. What are the implications for the indifference curves of an individual if he or she has *quasi-linear* preferences?

2. Can you think of a good for which *your* preferences are quasi-linear – that is, for which the reservation prices you would pay (or accept) are independent of the amount of money income that you have?
(3) If not, do you think that the reservation prices increase or decrease with your income? What would that imply about the shapes of your indifference curves?

(4) If indifference curves are not parallel, there is no longer a unique measure of how much better off is the individual when he or she moves from one point to another. Can you think of any approximate measures? (This is a difficult question and one you might not be able to answer until Chapter 19.)

3.11: Mathematical Appendix

Let me repeat the warning I have given earlier: if you have followed, and are happy with the results presented in the text, and do not like mathematics, then you do not need to read or understand this appendix. But if you like mathematics and are curious about the proof, you might find this appendix interesting. At a first glance it looks quite complicated, but when you understand what is going on, it is really quite simple.

The general proofs of the propositions in the text are straightforward. Assume quasi-linear preferences and that the individual starts with \( Q \) units of the good and \( M \) in money. We will consider only the situation when the price is sufficiently low for the individual to act as a buyer. The case when the price is high enough for the individual to act as a seller follows exactly the same line of argument.

Let us suppose that his or her reservation prices as a buyer are \( r_1, r_2, \ldots \) and so on. That is the individual would pay at most \( r_1 \) for the first extra unit of the good, \( r_2 \) for the second extra unit of the good, and so on. Let \( p \) denote the price of the good. Assume that \( p \) is less than \( r_1 \) – so the individual will buy at least one extra unit of the good. If \( p \) is greater than \( r_1 \) the individual will not want to be a buyer – the price is too high even to buy one unit of the good.
The individual starts with an endowment of \((Q,M)\) that is, \(Q\) units of the good and \(M\) in money. If he or she buys one unit of the good at the price of \(p\), he or she will have \((Q+1,M-p)\) - that is, one more unit of the good and \(p\) less in money; if he or she buys two units of the good, then he or she will have \((Q+2,M-2p)\) - that is 2 more units of the good and \(2p\) less in money; and so on. We can therefore construct the third column of the table below, in which the first column is the amount of the good after any purchases.

<table>
<thead>
<tr>
<th>quantity of the good</th>
<th>quantity of money on original indifference curve</th>
<th>quantity of money at a price of (p)</th>
<th>How much better off is the individual? This is given by the difference between the third and the second columns</th>
<th>Change in fourth column from row before</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Q)</td>
<td>(M)</td>
<td>(M)</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>(Q+1)</td>
<td>(M - r_1)</td>
<td>(M - p)</td>
<td>(r_1 - p)</td>
<td>(r_1 - p)</td>
</tr>
<tr>
<td>(Q+2)</td>
<td>(M - r_1 - r_2)</td>
<td>(M - 2p)</td>
<td>(r_1 + r_2 - 2p)</td>
<td>(r_2 - p)</td>
</tr>
<tr>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
</tr>
<tr>
<td>(Q+i)</td>
<td>(M - r_1 - r_2 - \ldots - r_i)</td>
<td>(M - ip)</td>
<td>(r_1 + r_2 + \ldots + r_i - ip)</td>
<td>(r_i - p)</td>
</tr>
<tr>
<td>(Q+i+1)</td>
<td>(M - r_1 - r_2 - \ldots - r_{i+1})</td>
<td>(M - (i+1)p)</td>
<td>(r_1 + r_2 + \ldots + r_{i+1} - (i+1)p)</td>
<td>(r_{i+1} - p)</td>
</tr>
<tr>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
</tr>
<tr>
<td>(Q+n)</td>
<td>(M - r_1 - r_2 - \ldots - r_n)</td>
<td>(M - np)</td>
<td>(r_1 + r_2 + \ldots + r_n - np)</td>
<td>(r_n - p)</td>
</tr>
</tbody>
</table>

The second column of the table indicates the quantity of money on the original indifference curve at the various amounts of the good. The first row indicates the starting position – if the individual stays there he or she is no better off than originally. This implies a zero in the fourth column - which is calculated as the difference between the third and second columns.
The second row of the table indicates what happens if the individual buys one unit of the good at a price of $p$ – he or she will then have $q+1$ units of the good and $m-p$ in money. How much better off is the individual if he or she does this? As we discuss in the text, with quasi-linear preferences we can measure this by how much vertically higher the individual is than initially. We can calculate this exactly as we know where the original indifference curve is at a quantity of $q+1$ – it is at a quantity of money $m - r_1$. Why? Because $r_1$ is the reservation price for the first extra unit bought and therefore is the maximum that the individual would pay to have $Q+1$ units of the good. With the price of $p$ the individual could be at $(Q+1,Q-p)$ which is higher by $r_1 - p$ than the original indifference curve. So the individual is $r_1 - p$ better off than originally by buying one unit at a price of $p$. We put this in the fourth column of the table. This of course is what we have claimed in the text. Note that we have assumed that $p$ is less than $r_1$ so this is a positive improvement.

The third row of the table indicates what would happen if the individual bought 2 extra units. He or she would have $Q+2$ of the good and $M-2p$ in money. How much higher is this than the original indifference curve? We know that this curve passes through the point $(Q+2,M-r_1-r_2)$ because $r_1$ and $r_2$ are the reservation prices for the first two units of the good. Hence the individual would be $(M-2p) - (M-r_1-r_2)$ higher than originally – that is an amount $r_1 + r_2 - 2p$ better off than originally. We can continue in this way, and can therefore get the general case – presented in the final row of the table. Here the individual buys $n$ extra units and ends up $r_1 + r_2 + ... + r_n - np$ better off.

The question now is: what is the best thing for the individual to do? Obviously this depends upon the price and the reservation prices of the individual. We know that in general these reservation prices are decreasing – that is, the individual would be happy to pay more for the first extra unit than for the second, and more for the second than for the third, and so on. In general, therefore, $r_1 > r_2 > ... > r_n > ...$ It therefore follows that $r_i - p$ must be decreasing and will become negative at some point. Let us define $i$ as the number such that $r_i - p$ is positive and that $r_{i+1} - p$ is negative.
(Obviously this \( i \) depends upon \( p \).) Now look at the fourth column of the table above. From the first row to the second row the difference is \( r_1 - p \); from the second row to the third row the difference is \( r_2 - p \); and in general the difference between the \( i \)'th row and the \((i+1)\)'th row is \( r_{i+1} - p \). We put these differences in the fifth column of the table. So the entries in the fourth column are increasing until we get to the \( i \)'th row, where \( i \) is defined above. It follows immediately that to buy \( i \) units of the good is the best thing that the individual can do, as this maximises how much better off is the individual. So at a price \( p \) such that \( r_i - p \) is positive and that \( r_{i+1} - p \) is negative the optimal demand is \( i \) units of the good. This simply states that the price is such that the individual would be happy to buy up to and including \( i \) units of the good but would not be willing to buy the \((i+1)\)th extra unit – because the price is higher than his or her reservation price for the \((i+1)\)th extra unit.

The condition that that \( r_i - p \) is positive and that \( r_{i+1} - p \) is negative can be re-written \( r_i > p > r_{i+1} \); and hence we have the result that for a price in this range the optimal demand is \( i \) units of the good.

Furthermore we see immediately from the above table that the surplus of the individual at such a price is \( r_1 + r_2 + ... + r_i - ip \) – he or she is that much better off than originally.

This is all we need. For any price \( p \) we can find the corresponding \( i \) and hence find the demand. It immediately follows that the demand is 1 unit for \( r_1 > p > r_2 \), is 2 units for \( r_2 > p > r_3 \), is 3 units for \( r_3 > p > r_4 \), and, in general, is \( i \) units for \( r_i > p > r_{i+1} \). It is therefore a step function as we have claimed in the text, with steps at the reservation prices. Moreover the surplus at a price of \( p \) such that \( r_i > p > r_{i+1} \) is exactly \( r_1 + r_2 + ... + r_i - ip \) as claimed in the text. This is precisely the area between the price of \( p \) and the step demand function we have derived – the sum of \( i \) rectangles each of width 1 unit.