Preferences Over Sets of Lotteries

WOJCIECH OLSZEWSKI
Northwestern University

First version received May 2004; final version accepted June 2006 (Eds.)

1. INTRODUCTION

In the present paper, I explore a model of decision-making under uncertainty, dispensing with state space. In period 1, a decision-maker chooses a set of (objective) lotteries. In period 2, Nature chooses a lottery from the set chosen by the decision-maker, and the decision-maker consumes the lottery chosen by Nature. For example, an agent who buys a lottery ticket may know that the prize of $2 is no more likely than the prize of $1 and the prize of $1 is no more likely than the prize of $0, but she may not know the exact probability distribution over the ticket’s prize; then the lottery ticket can be represented by the set of all probability distributions \( \pi \) such that \( \pi(\$2) \leq \pi(\$1) \leq \pi(\$0) \).

The model offers a natural concept of objective ambiguity and a decision-maker’s attitude to this sort of ambiguity. Indeed, larger sets of lotteries can be interpreted as more ambiguous objective information. An observation that a decision-maker preferred a smaller set to a larger set does not yet mean that the decision-maker is objective ambiguity averse because the larger set could be larger just as a result of “bad” lotteries. In general, the present model does not provide any concept of absolute attitude to objective ambiguity. However, it does provide a concept of comparative attitude to objective ambiguity: if decision-maker 1 prefers a set \( A \) over a set \( B \supset A \) when decision-maker 2 prefers \( A \) over \( B \) and the preferences of the two decision-makers coincide on the single lotteries from set \( B \), then I will say that decision-maker 1 is more objective ambiguity averse than decision-maker 2.

A variation of the famous Ellsberg paradox can serve as an illustration. Suppose that a decision-maker can choose between two urns: one urn contains 49% winning and 51% losing tickets and the other urn contains an unspecified assortment of winning and losing tickets. The decision problem can be interpreted as a choice between the single lottery \( l \) that yields a prize with probability 0.49 on the one hand and the set of all lotteries \( L = \{ l_p : p \in [0, 1] \} \), where \( l_p \) yields the prize with probability \( p \), on the other. The Ellsberg paradox suggests that most

1. The paper has been presented at Northwestern University, Princeton University, the University of Minnesota, the Canadian Economic Theory Conference in 2003, the ASSA Summer Meeting in 2003, and the Economic Theory Conference, Rhodes 2003.
decision-makers prefer the singleton \( \{l\} \); they seem to be more objective ambiguity averse than an individual who is indifferent between the set of lotteries \( L \) and the single lottery in which winning and losing are equally likely.

It is important to emphasize that I study decision problems in which it is unclear what the state space is, rather than a reduced form of decision problems in which the state space is disregarded. I argue that it may a good idea to dispense with state space in the former case, while in the latter case, there may be a loss entailed by avoiding a model with explicit state space. In addition, the current setting says nothing about ambiguity in the Ellsbergian interpretation of the term. The preferences studied in this paper do not necessarily involve behaviour that is demonstratively a departure from probabilistic sophistication.\(^2\)

I assume preferences over sets of lotteries, which can be represented by an expected utility over single lotteries; further, I impose axioms that generalize those often imposed on preferences over single lotteries in the existing literature. If Set S-Independence, Disjoint Set Betweenness (DSB), and Set Continuity are satisfied, then the decision-maker evaluates sets of lotteries \( A \) according to the rule

\[
H_{\alpha}(A) = \alpha \left( \max_{l \in A} U(l) \right) + (1 - \alpha) \left( \min_{l \in A} U(l) \right),
\]

where \( U(l) \) is the expected utility of lottery \( l \). The parameter \( \alpha \) can be interpreted as a measure of the decision-maker’s attitude to objective ambiguity. Indeed, it turns out that if \( \alpha_1 \) corresponds to a decision-maker \( i = 1, 2 \), then decision-maker 1 is more objective ambiguity averse than decision-maker 2 if and only if \( \alpha_1 \leq \alpha_2 \).

I argue, however, that the axiom of Set Continuity is too strong, in a relevant sense. If it is replaced with a weaker axiom, Set S-Solvability, and DSB is replaced with two stronger axioms, Generalized DSB (GDSB) and Two-Set Union, then the decision-maker still applies the rule (1) but only to the family of convex polyhedra. The present paper belongs to the large literature on decision-making under complete (or extreme) uncertainty, dating from Arrow and Hurwicz (1972), which studies preferences over sets of objects, that is, the elements of the power set \( 2^X \), given a preference over single objects, that is, the elements of a set \( X \). Sets of objects are interpreted as in the present paper, that is, the decision-maker has no information as to which element of the set that she chooses she will finally have to consume. Bossert, Pattanaik and Xu (2000) assume axioms that reflect some sort of attitude to ambiguity; more precisely, their Simple Uncertainty Aversion that says \( \{x\} \prec \{y\} \prec \{z\} \) implies \( \{x, z\} \prec \{y\} \). Barberà, Bossert and Pattanaik (2001) survey the results of this literature and Bossert (1997) provides a discussion of uncertainty aversion in non-probabilistic models.\(^3\)

In the literature, it is assumed that \( X \) is a finite set equipped with no structure other than a ranking of its objects. The innovation of the present paper is that sets of lotteries over objects, rather than sets of objects themselves, are examined. This additional structure allows a measurement of (comparative) attitude to objective ambiguity. In Section 5, I provide more details on the relation between my axioms and results and those from the existing literature.

This paper can also be viewed as an attempt to provide an alternative method for studying ambiguity to that developed within the Savage-type models by Gilboa and Schmeidler (1989), Schmeidler (1989), and many others. There seems to be no clear way of relating the two

---

2. I would like to thank a referee and the editor for a helpful discussion regarding this issue.

3. Another paper that characterizes preferences over sets of objects, uses betweenness axioms, and derives decision rules that involve only the maximal and minimal elements is Spiegler (2001). However, he assumes a different interpretation of preferences than the literature on complete uncertainty.
approaches, but there is probably a common feeling that they capture a slightly different notion of ambiguity.\footnote{It should be mentioned here that Ghirardato, Maccheroni and Marinacci (2004) (see also Jaffray, 1989; Ghirardato, 2001) provide axioms characterizing the counterpart of \( H_{0,a} \) in the Savage setting. More precisely, they consider preferences that evaluate acts, that is, mappings \( f : S \to X \) from the state space to the outcome space, according to a weighted average of the maximum and minimum expected utility with respect to a set of priors \( C \) over \( S \). They show that this counterpart of \( H_{0,a} \) can be characterized by an axiom that requires the decision-maker to be indifferent between acts that induce the same interval of expected utilities when they are ranged over the elements of \( C \).}

Another line of economics research related to this paper is developed in Kreps (1979), Dekel, Lipman and Rustichini (2001, 2004), and Gul and Pesendorfer (2001). They also characterize preferences over sets of lotteries, but interpret sets of lotteries differently. However, despite the different interpretations, there are similarities between their axioms and those postulated in this paper. Remark 1 from Section 3 contains a detailed discussion of these differences and similarities.

A number of other, closely related studies should also be mentioned. First, Wang (2003) studies preferences over pairs \( (A, l) \), where \( A \) is a set of lotteries and \( l \) is some “reference” lottery. A decision-maker who satisfies Wang’s axioms evaluates pairs \( (A, l) \) according to the minimum over the set of lotteries \( A \) of an expected utility plus a penalty function that penalizes deviation from the reference probability \( l \). Gajdos, Tallon and Vergnaut (2005) study preferences over pairs \( (f, A) \), where \( f : S \to X \) is an act, and \( A \) is a set of probability distributions over the set \( S \); they interpret \( A \) as the decision-maker’s objective information about the state of the world. A decision-maker who satisfies their axioms evaluates pairs \( (f, A) \) according to the minimum of the expected utilities of \( f \) over a set of probability distributions contained in the closure of the convex hull of \( A \). Finally, Ahn (2003) studies the same model as here. It defines the (comparative) attitude to ambiguity in a manner similar to that in the present paper, characterizes axiomatically a different family of decision rules, and (within that family) it provides a measure of attitude to (what I call objective) ambiguity. Section 5 contains a slightly more detailed comparison of the two papers.

2. MODEL

Let \( X = \{x_1, \ldots, x_n\} \) be a finite set of outcomes, and let \( \Delta = \Delta(X) \) be the set of all probability distributions, or lotteries, over \( X \) endowed with the standard topology. I denote by \( x \) a generic element of \( X \) and by \( l \) a generic element of \( \Delta \). The outcome \( x \) is sometimes identified with the lottery that yields \( x \) with probability 1. I write \( l = (q_1, \ldots, q_n) \), when \( l \) yields \( x_i \) with probability \( q_i \) for \( i = 1, \ldots, n \). Compound lotteries are identified with their reduced-form simple lotteries.

Let \( 2^\Delta \) stand for the set of all non-empty subsets of \( \Delta \) and let \( A \subset 2^\Delta \) denote the set of all closed and non-empty subsets of \( \Delta \); finally, let \( V \subset A \) stand for the set of all polyhedra contained in \( \Delta \), where by a \emph{polyhedron} I understand the union of a finite family of geometric simplices, possibly of different dimensions. I denote by \( A, B \) generic elements of \( 2^\Delta \). The lottery \( l \) is sometimes identified with the singleton \( \{l\} \). Given a set \( A \in V \), let \( \text{bd}A \) and \( \text{int}A \) denote, respectively, the geometric boundary and interior of \( A \); let \( \text{Co}A \) stand for the convex hull of \( A \).

I study transitive and complete binary relations \( \preceq \) defined over a family of subsets of \( 2^\Delta \) containing all singletons \( \{l\} \), such as \( A \) or \( V \). I assume that \( x_1 \preceq \cdots \preceq x_n \) and \( x_1 < x_n \). I also assume that the relation \( \preceq \) restricted to single lotteries admits an expected utility representation. Given a utility index \( u : X \to R \) and a lottery \( l = (q_1, \ldots, q_n) \), let

\[
U(l) = \sum_{i=1}^n q_i u(x_i),
\]

denote the expected utility of the lottery \( l \).
2.1. Compound sets of lotteries

Most of the axioms that I impose on sets of lotteries will refer to compound sets of lotteries. Given a set \( P \subseteq [0, 1] \), define \( PA_1 + (1 - P)A_2 \) as the set \( \{l = pl_1 + (1 - p)l_2 : l_1 \in A_1, l_2 \in A_2, p \in P\} \). Notice that (1) if \( A_1, A_2 \) are closed subsets of \( \Delta \) and \( P \) is a closed subset of \([0, 1]\), then \( PA_1 + (1 - P)A_2 \) is a closed subset of \( \Delta \); (2) if \( A_1, A_2 \), and \( P \) are polyhedra, then \( PA_1 + (1 - P)A_2 \) is a polyhedron. If \( P = \{p\} \) is a singleton, I write \( pA_1 + (1 - p)A_2 \) instead of \( PA_1 + (1 - P)A_2 \).

I say that a triple of sets \( A_1, A_2, \) and \( P \) has the uniqueness property if for every \( l \in PA_1 + (1 - P)A_2 \) there are unique \( l_1 \in A_1, l_2 \in A_2, p \in P \) such that \( l = pl_1 + (1 - p)l_2 \), except the multiplicity of \( l_2 \) in the representation \( l_1 = l_1l_1 + 0l_2 \) of lotteries \( l_1 \in A_1 \) and the multiplicity of \( l_1 \) in the representation \( l_2 = 0l_1 + l_2 \) of lotteries \( l_2 \in A_2 \).

Motivating my axioms, I assume that the decision-maker identifies (or is indifferent between) the set \( PA_1 + (1 - P)A_2 \) and the situation in which the lottery \( l \) that she will consume is chosen in the following two-step procedure: (1) Nature draws \( A_1 \) with probability \( p \) and \( A_2 \) with probability \( 1 - p \), and the decision-maker knows only that \( p \in P \); (2) Nature draws \( l \) from the selected \( A_1 \).

I comment on this identification assumption later, when introducing my axioms. For now, note only that the uniqueness property guarantees that the correspondence between the lotteries from \( PA_1 + (1 - P)A_2 \) and the compound lotteries from the two-step procedure is 1:1. This seems important for my identification assumption to be well motivated. For example, suppose that \( A_1 = A_2 = [0, 1] \) and \( P = \{1/2\} \), where 0 means $0 for sure and 1 means $1 for sure. Then the set of compound lotteries from the two-step procedure consists of four elements, while the set
\[
PA_1 + (1 - P)A_2
\]
consists only of three elements, as \((1/2)0 + (1/2)1 = (1/2)1 + (1/2)0\). Thus, by assuming that the decision-maker is indifferent between the set \( PA_1 + (1 - P)A_2 \) and the situation in which the lottery that she will consume is chosen in the two-step procedure, I would make the decision-maker disregard the information that one of the lotteries in the set \( PA_1 + (1 - P)A_2 \) is the reduced form for two different compound lotteries from the two-step procedure. However, this information may affect the decision-maker’s preference, as she may (subjectively) anticipate that the reduced forms of multiple compound lotteries are more likely to be the lottery that she will have to consume.

2.2. Set S-Solvability and Set S-Independence

Throughout the paper, I impose the following axioms:

**Axiom 1 (Set S-Solvability).** *For all triples of sets \( A, A_1, A_2 \) such that one of the sets \( A_1, A_2 \) is a singleton, if \( A_1 \preceq A \preceq A_2 \), then there exists a \( p \in [0, 1] \) such that \( A \sim pA_1 + (1 - p)A_2 \).*

This axiom is a combination of two assumptions. First, the decision-maker identifies sets \( pA_1 + (1 - p)A_2 \) with the compound lotteries from the two-step procedure described in Section 2.1; this assumption is weaker here than in general, as both \( P \) and one of the sets \( A_1, A_2 \) are singletons; in particular, the triple \( A_1, A_2, \) and \( P \) always has the uniqueness property. Second, the preference has the mean-value property with respect to \( p \) used in the first step of the two-step procedure. Set S-Solvability imposes on preferences some sort of continuity. However, I argue below that in the present context, Set S-Solvability is a better axiom than is continuity with respect to the Hausdorff metric.
Axiom 2 (Set S-Independence). If $A_1 \precsim A_2$ and $A = \{l\}$ is a singleton, then $pA_1 + (1-p)A \precsim pA_2 + (1-p)A$ for every $p \in [0,1]$.

This axiom is again a combination of two assumptions. First, the decision-maker identifies sets $pA_1 + (1-p)A_2$ with the compound lotteries from the two-step procedure described in Section 2.1. Second, the preference satisfies the traditional Weak Independence axiom with respect to mixing in the first step of the two-step procedure.

Set S-Independence generalizes traditional Weak Independence over single lotteries and Set S-Solvability generalizes Solvability over single lotteries (see Dekel, 1986). Independence (both Strict and Weak) and Solvability imply the expected utility representation. However, Strict Independence over single lotteries is not implied by my Axioms 1 and 2; it is, therefore, imposed on top of the two axioms by the assumption that the relation $\precsim$ restricted to single lotteries admits an expected utility representation.

3. SET CONTINUITY AND DSB

Consider the space of all closed subsets of $\Delta$, equipped with the Hausdorff distance

$$\text{dist}(A, B) = \max \left\{ \max_{a \in A} \min_{b \in B} \text{dist}(a, b), \max_{b \in B} \min_{a \in A} \text{dist}(a, b) \right\} .$$

See Engelking (1989) for more details on the Hausdorff distance.

Axiom 3 (Set Continuity). For every $A_1$, sets $\{A_2 : A_2 \prec A_1\}$ and $\{A_2 : A_1 \prec A_2\}$ are open in the space of all closed subsets of $\Delta$ equipped with the Hausdorff distance.

This axiom generalizes traditional continuity for single lotteries and can be interpreted in a similar manner. However, I argue below that Set Continuity is a rather strong axiom, more demanding than traditional continuity. I relax this axiom in Section 4, and I (partially) generalize the representation theorem from this section to preferences that violate Set Continuity. Notice that Set Continuity implies Set S-Solvability.

Axiom 4.

(a) (Weak DSB). For any disjoint sets $A_1, A_2 \subset \Delta$, if $A_1 \precsim A_2$, then $A_1 \precsim A_1 \cup A_2 \precsim A_2$.

(b) (Strict DSB). If $l_1 < l_2$ for all $l_1 \in A_1$ and $l_2 \in A_2$, then $A_1 < A_1 \cup A_2 < A_2$.

I call Weak DSB and Strict DSB together DSB. Weak DSB seems to be an intuitive property: it says that if the set $A_1$ is “bad” and the set $A_2$ is “good”, the union should be somewhere in between or that the decision-maker cannot be made worse off (respectively better off) by a chance that the lottery that she will have to consume is chosen from a “good” set instead of a “bad” set (respectively from a “bad” set instead of a “good” set).

The requirement that sets $A_1$ and $A_2$ are disjoint seems important for Axiom 4 to be well motivated. Otherwise, “good” lotteries from $A_2$ could have already belonged to $A_1$, while “bad” lotteries could not have, and then it would not necessarily be intuitive to assume that $A_1 \precsim A_1 \cup A_2$. For example, suppose that both $A_1$ and $A_2$ consist of two lotteries: respectively, $0$ for sure and $1$ for sure and $0 \cdot 1$ for sure and $1$ for sure. It would not be unreasonable to strictly

5. It is easy to see that the union of two sets coincides with the operation $PA_1 + (1-P)A_2$, defined in Section 2.1, for $P = [0,1]$, and the assumption that the two sets are disjoint corresponds to the uniqueness property.

© 2007 The Review of Economic Studies Limited
prefer both $A_1$ and $A_2$ to $A_1 \cup A_2$ because each of them contains just one “bad” lottery and $A_1 \cup A_2$ contains two “bad” lotteries.

The requirement imposed on $\prec$ by the Strict DSB axiom is analogous to, but slightly weaker than, that imposed on $\lesssim$ by Weak DSB. Namely, it requires that $l_1 \prec l_2$ for all $l_1 \in A_1, l_2 \in A_2$, not only that $A_1 \prec A_2$. To see the motivation, consider the following example. Let $A_1 = \{l_0, l_1\}$, where $l_0$ yields $0$ for sure and $l_1$ yields $1$ for sure, and let $A_2$ consist of a single lottery $l'_1$ such that the decision-maker is indifferent between $l'_1$ and $l_1$; say, it is the equivalent of $1$ in another currency. It is reasonable to expect that $A_1 \prec A_2$. If the decision-maker believes that Nature picks $l_1$ or $l'_1$ first and then (if $l_1$ gets picked) it decides between $l_1$ and $l_0$, then it is also reasonable to expect $A_1 \prec A_1 \cup A_2$. If, however, the decision-maker believes that Nature picks $\{l_1, l'_1\}$ or $\{l_0\}$ first and then (if $\{l_1, l'_1\}$ gets picked) it decides between $l_1$ and $l'_1$, it is not unreasonable for that decision-maker to be indifferent between $A_1 \cup A_2$ and $A_1$. More generally, there sometimes exist multiple representations of a certain set as the union of two disjoint sets, which suggest different conclusions regarding strict preference. However, it is easy to see that if $l_1 \prec l_2$ for all $l_1 \in A_1, l_2 \in A_2$, no representation of $A_1 \cup A_2$ as the union of two disjoint sets suggests that $A_1 \cup A_2 \sim A_1$ or $A_1 \cup A_2 \sim A_2$.

3.1. Representation theorem

Recall the rule (1) and define the preference $\sim^{H_{u,\alpha}}$ on $\mathcal{A}$, the family of all closed subsets of the simplex $\Delta$, by

$$A_1 \sim^{H_{u,\alpha}} A_2 \text{ if and only if } H_{u,\alpha}(A_1) \leq H_{u,\alpha}(A_2).$$

**Theorem 1.** A preference $\prec$ defined on $\mathcal{A}$ satisfies Set Continuity, Set S-Independence, and DSB if and only if there exists a unique $\alpha \in (0, 1)$ such that

$$\prec \equiv \sim^{H_{u,\alpha}},$$

where $u : X \rightarrow \mathbb{R}$ is a utility index of outcomes. The parameter $\alpha$ remains unaltered if the utility index $u$ gets replaced with its positive affine transformation.

**Proof.** See Appendix A. \hfill ||

To prove Theorem 1, I first characterize the indifference curves on the two-lottery sets of the form $\{ax_n + (1 - a)x_1, bx_n + (1 - b)x_1\}$. The sets of this form can be identified with plane points $(a, b)$ belonging to the triangle $\text{Co}(0, 0), (0, 1), (1, 1)$ (see Figure 1). Theorem 1 (restricted to the two-lottery sets) is an analytical counterpart of the statement that for any $\prec$ that satisfies my three axioms, the indifference curves are straight and parallel, downward-sloping lines joining the sides $\text{Co}(0, 0), (0, 1)$ and $\text{Co}(0, 1), (1, 1)$ with $\text{Co}(0, 0), (1, 1)$. Strict DSB implies that $(1, 1)$ is strictly preferred to $(0, 1)$ and $(0, 1)$ is strictly preferred to $(0, 0)$. Thus, by Set S-Solvability, there is a point $(a, a) \in \text{Co}(0, 0), (1, 1)$ such that $(0, 1)$ and $(a, a)$ belong to the same indifference curve. This in turn implies, by Set S-Independence, that the straight line joining the points $(0, 1)$ and $(a, a)$ is an indifference curve. Again by Set S-Independence, it may be seen that the straight lines parallel to this line are the indifference curves.

Next, I show (the argument is omitted in this sketch of the proof) that the preference ranking of a two-lottery set depends only on the ranking of the two lotteries; that is, if $l_1 \sim ax_n + (1 - a)x_1$,

6. An interested reader will notice that if $l_1 \prec l_2$ for all $l_1 \in A_1, l_2 \in A_2$ in Strict DSB gets replaced with $A_1 \prec A_2$, then my representation theorems become impossibility results, in the sense that no preference satisfies the required axioms.
and \( l_2 \sim b x_n + (1 - b) x_1 \), then

\[
\{ l_1, l_2 \} \sim \{ a x_n + (1 - a) x_1, b x_n + (1 - b) x_1 \}.
\]

The rest of the proof consists of two steps. First, I show that

\[
\{ l_1, l_3 \} \sim \{ l_1, l_2, l_3 \},
\]

for any triple of lotteries \( l_1 \not\simeq l_2 \not\simeq l_3 \). This follows from Weak DSB, Set Continuity, and the characterization of preferences over two-lottery sets. Indeed, by this characterization, \( \{ l_1, l_2 \} \not\simeq \{ l_1, l_3 \} \not\simeq \{ l_2, l_3 \} \). If Weak DSB held not only for disjoint sets, but for any sets, it would imply that \( \{ l_1, l_2 \} \not\simeq \{ l_1, l_2, l_3 \} \not\simeq \{ l_1, l_3 \} \) and \( \{ l_1, l_3 \} \not\simeq \{ l_1, l_2, l_3 \} \not\simeq \{ l_2, l_3 \} \). Although Weak DSB holds only for disjoint sets, a similar argument still applies due to Set Continuity. Indeed, \( \{ l_1, l_2, l_3 \} \) can be represented as the limit of \( \{ l_1, l_2 \} \cup \{ l_1^n, l_3 \} \), where \( l_1 \neq l_1^n \to l_1 \), as well as the limit of \( \{ l_1^n, l_3 \} \cup \{ l_2, l_3 \} \), where \( l_3 \neq l_3^n \to l_3 \).

Finally, I show that any closed subset \( A \subset \Delta \) is the limit of the unions of finite families of three-lottery sets such that the best lottery in each of the three-lottery sets converges to an \( l_{\text{max}} \in A \) with \( U(l_{\text{max}}) = \max_{l \in A} U(l) \) and the worst lottery in each of the three-lottery sets converges to an \( l_{\text{min}} \in A \) with \( U(l_{\text{min}}) = \min_{l \in A} U(l) \). This yields, again by Weak DSB and Set Continuity, that

\[
A \sim \{ l_{\text{min}}, l_{\text{max}} \}.
\]

**Remark 1.** Kreps (1979), Dekel \textit{et al.} (2001, 2004), and Gul and Pesendorfer (2001) also study preferences over sets of lotteries, but with a different interpretation. As in the present paper, the decision-maker chooses a set of lotteries in period 1, but it is also the decision-maker (not Nature) who chooses a lottery from the set in period 2. Under this interpretation, sets of
lotteries can be used to capture a preference for flexibility (Kreps, 1979; Dekel et al., 2001) and temptation-driven preferences (Gul and Pesendorfer, 2001; Dekel et al., 2004).

Despite different interpretations, the methodological relationship between this paper and the results reported in the papers just mentioned is quite strong. Dekel et al. (2001) assume Set Continuity and Set Independence, that is, a slightly stronger axiom than my Set S-Independence that does not require the set $A$ to be a singleton; they also assume Convexity that says that the decision-maker is indifferent between a set and its convex hull. They show that the decision-makers who satisfy their axioms evaluate sets of lotteries $A$ according to the rule

$$
\int_S \sup_{l \in A} U(l, s) \mu(ds),
$$

where $S$ is a state space, $\mu$ is a finitely additive (not necessarily positive) measure on $S$, and $U(l, s)$ is the expected utility of lottery $l$ in state $s$.

Gul and Pesendorfer (2001) assume additionally Set Betweenness, that is, a slightly stronger axiom than my Weak DSB that does not require the sets $A_1$ and $A_2$ to be disjoint. They show that the Dekel et al. representation when combined with this additional axiom yields the result that decision-makers evaluate sets of lotteries $A$ according to the rule

$$
\sup_{l \in A_1} U_1(l) - \sup_{l \in A_2} U_2(l),
$$

which is Dekel et al.’s representation with two states and the measure that assigns 1 to one of them and $-1$ to the other.

The rule (1) is a refinement of Gul and Pesendorfer’s representation, in which $U_1(l) = \alpha U(l)$ and $U_2(l) = -(1 - \alpha)U(l)$. Their representation refines to (1) due to Strict DSB, as the other axioms from Theorem 1 are implied immediately by those imposed by Gul and Pesendorfer; in Example 3(b), I actually show directly that (1) is the only rule among those characterized by Gul and Pesendorfer that satisfies Strict DSB. One can actually criticize Gul and Pesendorfer’s, as well as Dekel et al.’s (2004), representations on these grounds, as it seems that many decision-makers with temptation-driven preferences would satisfy Strict DSB, yet their preferences need not be represented by (1).

Finally, it should be emphasized that Gul and Pesendorfer’s and Dekel et al.’s justification of Set Independence, as well as other axioms involving compound sets of lotteries, differs from that behind Set S-Independence and other axioms involving the sets $P A_1 + (1 - P)A_2$ imposed in this paper. Gul and Pesendorfer and Dekel et al. justify their axiom as a combination of an ordinary independence argument (applied to sets of lotteries) and the assumption that the decision-maker is indifferent as to the timing of the resolution of uncertainty, that is, she is indifferent whether the uncertainty regarding $A_1$ or $A_2$ is resolved before or after she decides which lottery to choose from each of the two sets.

3.2. Measure of (comparative) attitude to objective ambiguity

In the present setting, I define objective ambiguity as the lack of a single (objective) probability distribution over outcomes, and I interpret larger sets of possible probability distributions as more
ambiguously ambiguous information. Given preferences $\succeq_1$ and $\succeq_2$, I call congruent any set $B \subset \Delta$ such that $\succeq_1$ and $\succeq_2$ coincide on single lotteries from the set $B$. If the following two implications hold for any congruent set $B$ and its subset $A \subset B$:

$$B \succeq_2 A \Rightarrow B \succeq_1 A \quad \text{and} \quad B \succeq_1 A \Rightarrow B \succeq_2 A,$$

then I say that the preference $\succeq_1$ is more objective ambiguity averse than the preference $\succeq_2$, or that a decision-maker with the preference $\succeq_1$ is more objective ambiguity averse than a decision-maker with the preference $\succeq_2$.

It is important, from a conceptual perspective, to restrict attention to congruent sets in the definition of what it is to be more objective ambiguity averse. Otherwise, reasons that, intuitively, have nothing to do with the attitude to objective ambiguity might affect differences in the preferences $\succeq_1$ and $\succeq_2$ over sets $A \subset B$. Consider an extreme example, in which decision-maker 1 prefers any lottery from $B - A$ to any lottery from $A$, while decision-maker 2 prefers any lottery from $A$ to any lottery from $B - A$. Then decision-maker 1 prefers $B$ to $A$, while decision-maker 2 prefers $A$ to $B$. However, this has nothing to do with the intuition regarding their attitude to objective ambiguity.

Note that the definition of comparative attitude to objective ambiguity is purely behavioural, in the sense that to decide which of the two decision-makers is more objective ambiguity averse, it is necessary only to observe their preferences. It follows from Theorem 1 that if a preference satisfies the axioms imposed there, then the attitude to objective ambiguity is characterized by the parameter $\alpha$.

**Corollary 2.** Suppose that $\succeq_1$ and $\succeq_2$ defined on the family $A$ of all closed subsets of $\Delta$ satisfy Set Continuity, Set $S$-Independence, and DSB. Then the preference $\succeq_1$ is more ambiguity averse than the preference $\succeq_2$ if and only if $\alpha_1 \leq \alpha_2$.

**Proof.** Consider first the family of subintervals of the interval $\text{Co}(x_1, x_n)$; these subintervals can be identified with the plane points belonging to the triangle from Figure 1. I shall show that the condition (2) is satisfied for any such pair of subintervals $A \subset B$ if and only if $\alpha_1 \leq \alpha_2$.

Indeed, if $A \subset B$ then the plane point $\overline{B}$ representing $B$ in Figure 1 lies to the north-west of the plane point $\overline{A}$ representing $A$. If $\overline{B}$ lies below the indifference curve of $\succeq_2$ passing through $\overline{A}$, then it must also lie below the indifference curve of $\succeq_1$ passing through $\overline{A}$ whenever the latter indifference curve is weakly steeper than the former. This yields the sufficient condition.

Conversely, every plane point that lies to the north-west of $\overline{A}$ representing some $A$ represents some $B$ such that $A \subset B$. If the indifference curve of $\succeq_1$ passing through $\overline{A}$ is strictly flatter than the indifference curve of $\succeq_2$ passing through $\overline{A}$ and $\overline{A}$ is an interior point of the triangle, then there must be a point to the north-west of $\overline{A}$ that lies below the indifference curve of $\succeq_2$ and above the indifference curve of $\succeq_1$. This yields the necessary condition.

Thus, if $\succeq_1$ is more objective ambiguity averse than the preference $\succeq_2$, then $\alpha_1 \leq \alpha_2$. To show the converse, it is also necessary to consider congruent sets $B$ and their subsets $A$, which are not subintervals of the interval $\text{Co}(x_1, x_n)$. However, it is possible to apply the same argument as above to the interval $\text{Co}(l^B_{\min}, l^B_{\max})$ instead of $\text{Co}(x_1, x_n)$, where $U(l^B_{\min}) = \min_{l \in B} U(l)$ and $U(l^B_{\max}) = \max_{l \in B} U(l)$. Then one can consider the interval $\text{Co}(l^B_{\min}, l^B_{\max}) \sim B$ instead of $B$ and the interval $\text{Co}(l^A_{\min}, l^A_{\max}) \sim A$ instead of $A$, where $l^A_{\min} = \min_{l \in A} U(l)$ and $l^A_{\max} = \max_{l \in A} U(l)$ and $U(l^A_{\min}) = \min_{l \in A} U(l)$ and $U(l^A_{\max}) = \max_{l \in A} U(l)$.

**Remark 2.** The relation of “being more objective ambiguity averse” resembles, and actually borrows the key idea from, Yaari (1969), who defines “being more risk averse” within the
Savage model by the following two implications for every constant act $x$ and every uncertain act $f$:

$$f \preceq_2 x \Rightarrow f \succeq_1 x \text{ and } f \prec_2 x \Rightarrow f \prec_1 x,$$

assuming that the decision-makers with both preferences have the same beliefs over the states of the world. Restricting attention in the condition (2) to sets $A$ that are singletons, one obtains an alternative definition of “being more objective ambiguity averse”. There seems to be no decisive argument as to which of the two definitions is more plausible. However, it is easy to see that the two definitions coincide within the class of preferences that satisfy Set Continuity, Set S-Independence, and DSB.

3.3. Tightness of axioms

I will now establish the mutual independence of the three axioms from Theorem 1.

**Example 1.** Take any decision rule involving only the minimal and maximal elements of a set whose indifference curves are (in terms of Figure 1) non-parallel, downward-sloping straight lines that join the sides $\text{Co}((0, 0), (0, 1))$ and $\text{Co}((0, 1), (1, 1))$ with $\text{Co}((0, 0), (1, 1))$. Since its indifference curves are not parallel, this decision rule violates Set S-Independence.

On the other hand, it satisfies all other axioms imposed in Theorem 1 (and, it turns out, also all other axioms imposed in this paper). Set Continuity is straightforward. To see Weak DSB consider any pair of sets $A_1 \preceq A_2$; let $a_i = \min_{l \in A_i} U(l)$ and $b_i = \max_{l \in A_i} U(l)$. If $a_1 < a_2$ and $b_1 < b_2$, then Weak DSB applied to $A_1$ and $A_2$ is satisfied because $a_1 = \min_{l \in A_1 \cup A_2} U(l)$, $b_2 = \max_{l \in A_1 \cup A_2} U(l)$, and $(a_1, b_2)$ lies between the indifference curves passing through $(a_1, b_1)$ and $(a_2, b_2)$, as all indifference curves are downward-sloping. If $a_2 < a_1$ and $b_1 < b_2$ (or $a_1 < a_2$ and $b_2 < b_1$), then Weak DSB applied to $A_1$ and $A_2$ is satisfied as $a_1 = \min_{l \in A_1 \cup A_2} U(l)$, $b_1 = \max_{l \in A_1 \cup A_2} U(l)$ (or $a_2 = \min_{l \in A_1 \cup A_2} U(l)$, $b_2 = \max_{l \in A_1 \cup A_2} U(l)$). To see Strict DSB note that if $a_1 < a_2$ and $b_1 < b_2$ (in particular if $b_1 < a_2$), then $(a_1, b_2)$ lies strictly between the indifference curves passing through $(a_1, b_1)$ and $(a_2, b_2)$, assuming that no indifference curve is horizontal or vertical.

**Example 2.** The lexicographic ordering that ranks sets of lotteries primarily according to rule (1) and secondarily according to the size of the interval $[\min_{l \in A} U(l), \max_{l \in A} U(l)]$ is an example of preference that violates Set S-Solvability (and so Set Continuity). To see this, take any pair of lotteries $l_1 < l_2$ and notice that $\{l_1\} < \{l_1, l_2\} < \{l_2\}$, but there is no $p \in [0, 1]$ such that

$$p\{l_1\} + (1-p)\{l_2\} \sim \{l_1, l_2\}.$$  

Arguments similar to those used in Example 1 show that this lexicographic ordering satisfies all other axioms imposed in Theorem 1 (and, it turns out, all other axioms imposed in this paper).

**Example 3.**

(a) Let $n = 2$. Fix any non-constant expected utility function $U : \Delta \to R$. Given a closed set $A \subset \Delta$ and a direction, that is, a vector $v$ from the unit circle $S^1 = \{v \in R^2 : \text{dist}(v, 0) = 1\}$, consider a lottery $l_A(v) \in \arg\max_{l \in A} l_0$, that is, the lottery $l_A(v)$ maximizes the distance from the origin (among the elements of the set $A$) in the direction of the vector $v$. The choice of the lottery $l_A(v)$ may not be unique for some directions $v$, but it can easily be checked that the set of such directions is at most countable for every closed set $A \subset \Delta$. Let

$$H(A) = \frac{1}{4} \min_{l \in A} U(l) + \frac{1}{4} \max_{l \in A} U(l) + \frac{1}{2} \int U(l_A(v)) d\nu,$$

© 2007 The Review of Economic Studies Limited
where I integrate with respect to the uniform probability distribution over the circle \( S^1 \). The function \( U(l_A(v)) \) is integrable because it is continuous on the set of directions \( v \) such that the choice of the lottery \( l_A(v) \) is unique.

Notice that \( H([l]) = U(l) \) for every lottery \( l \in \Delta \). To see that the rule \( H \) satisfies Set Continuity, observe that if the choice of \( l_A(v) \) is unique and the sets \( A_n \) converge to \( A \) in the Hausdorff metric, the lotteries \( l_{A_n}(v) \) must converge to \( l_A(v) \). Set S-Independence follows from the fact that

\[
l_{pA_1+(1-p)[l]}(v) = l_{A_1}(v) + (1-p)l,
\]

and Strict DSB follows from the fact that if \( l_1 \prec l_2 \) for all \( l_1 \in A_1 \) and \( l_2 \in A_2 \), then

\[
l_{A_1}(v) \leq l_{A_1 \cup A_2}(v) \leq l_{A_2}(v),
\]

for all directions \( v \) such that the choice of both \( l_{A_1}(v) \) and \( l_{A_2}(v) \) is unique.

To see that the rule \( H \) violates Weak DSB, consider a pair of disjoint sets \( A_1 = \{l_1, l_3\} \) and \( A_2 = \{l_2, l_4\} \) such that \( U(l_1) = U(l_2) < U(l_3) = U(l_4) \). It can easily be verified that

\[
H(A_1) = H(A_2) = \frac{1}{2} U(l_1) + \frac{1}{2} U(l_3),
\]

but if \( \text{dist}(l_1, l_2) < \text{dist}(l_3, l_4) \), then

\[
\int U(l_{A_1 \cup A_2}(v))d\nu > \frac{1}{2} U(l_1) + \frac{1}{2} U(l_3),
\]

so

\[
H(A_1 \cup A_2) > \frac{1}{2} U(l_1) + \frac{1}{2} U(l_3).
\]

(b) Consider any pair of non-constant expected utility functions \( U_1, U_2 : \Delta \to \mathbb{R} \), where \( \dim \Delta \geq 2 \). Let

\[
G(A) = \max_{l \in A} U_1(l) - \max_{l \in A} U_2(l).
\]

Gul and Pesendorfer (2001) show that \( G \) satisfies Set Continuity, Set S-Independence (it does, in fact, satisfy a stronger axiom, where the set \( A \) need not be a singleton), and Weak DSB.

It turns out that if the indifference curves of \( U_1 \) are not parallel to the indifference curves of \( U_2 \), \( G \) does not satisfy Strict DSB. Indeed, for any lottery \( l_1 \) such that

\[
l_1 \notin \arg \max_{l \in \Delta} U_1(l) \text{ and } l_1 \notin \arg \max_{l \in \Delta} U_2(l),
\]

take any \( l_2 \) such that

\[
U_1(l_2) = U_1(l_1) + \varepsilon \text{ and } U_2(l_2) = U_2(l_1) + 2\varepsilon;
\]

this is always possible for small enough \( \varepsilon \) whenever the indifference curves of \( U_1 \) are not parallel to the indifference curves of \( U_2 \).

Then, for \( A_1 = \{l_1\} \) and \( A_2 = \{l_2\} \),

\[
G(A_1 \cup A_2) = G(A_2) = G(A_1) - \varepsilon.
\]
I begin this section with an argument that Set Continuity is a strong axiom, which would probably be violated by many decision-makers. Imagine that individuals choose between the following two sets of lotteries: \( A_1 = \{x_1, x_2\} \) and \( A_2 = \{x_1, x_2, l\} \), where \( x_1 \) and \( x_2 \) yield for sure $0 and $1, respectively, and \( l \) gives $0 with probability 0.01 and $1 with probability 0.99. It is likely that many individuals would strictly prefer \( A_2 \) to \( A_1 \) because \( A_2 \) contains two “good” lotteries and \( A_1 \) contains just one “good” lottery. Let \( A_n = \{x_1, x_2, l^n\} \), where \( l^n \) gives $0 with probability \( 10^{-n} \) and $1 with probability \( 1 - 10^{-n} \). Presumably, \( A_1 \prec A_2 \prec A_3 \prec \cdots \) would be the ranking of many individuals. However, the sequence \( \{A_n : n = 2, 3, \ldots\} \) converges to the set \( A_1 \) in the Hausdorff metric.

Intuitively, the decision-maker may like the fact that each of the sets \( A_n \ (n = 2, 3, \ldots) \) contains two “good” lotteries, but in the limit the two “good” lotteries get squeezed and the set \( A_1 \) contains only one “good” lottery. Note that this criticism of Set Continuity does not apply to the “continuity” imposed by Set S-Solvability.

In this section, I dispense with Set Continuity, replacing it with Set S-Solvability. Strengthening simultaneously DSB, I show that Theorem 1 still holds for an important family of sets of lotteries, convex polyhedra.

**Axiom 5.**

(a) (Weak GDSB). For any triple of sets \( A_1, A_2, \) and \( P \subset [0, 1] \) with the uniqueness property, if \( A_1 \not\prec A_2 \), then \( A_1 \not\prec PA_1 + (1 - P)A_2 \not\prec A_2 \).

(b) (Strict GDSB). Assuming \( l_1 \prec l_2 \) for all \( l_1 \in A_1 \) and \( l_2 \in A_2 \), (a) if \( P \neq \{0\} \), then \( PA_1 + (1 - P)A_2 \prec A_2 \); (b) if \( P \neq \{1\} \), then \( A_1 \prec PA_1 + (1 - P)A_2 \).

I call Axiom 5 GDSB. GDSB can be interpreted as a composition of two assumptions. First, the decision-maker identifies the set \( PA_1 + (1 - P)A_2 \) with the compound lotteries from the two-step procedure described in Section 2.1. Second, the decision-maker cannot be made worse off (respectively better off) by a chance that the lottery that she will consume is chosen from a “good” set instead of a “bad” set (respectively, from a “bad” set instead of a “good” set), no matter what she knows about how likely this chance is.

Despite an analogous motivation to DSB, Axiom 5 is obviously more demanding. It is often much less obvious that a certain set coincides with the reduced-form simple lotteries of some compound set of lotteries. In consequence, GDSB excludes some natural preferences. One may argue that some decision-makers may strictly prefer \( PA_1 + (1 - P)A_2 \) to \( A_1 \) and \( A_2 \) because the randomization allows for some form of hedging against extreme outcomes. For example, let \( A_1 = \{l_1, l_2\} \) and \( A_2 = \{l_3, l_4\} \), where \( l_1 \sim l_3 \prec l_2 \sim l_4 \). Then, assuming that the uniqueness property is satisfied, \( (1/2)A_1 + (1/2)A_2 \) consists of four lotteries: \( (1/2)l_1 + (1/2)l_3 \sim l_3 \), \( (1/2)l_2 + (1/2)l_4 \sim l_4 \), and \( l_3 \prec (1/2)l_1 + (1/2)l_4 \prec (1/2)l_2 + (1/2)l_4 \). Some decision-makers may indeed feel safer when the set contains two lotteries of intermediate quality.

I show in Example 6 that preferences that are sensitive to the volume of lotteries with certain utility typically violate GDSB.

**Axiom 6 (Two-Set Union).**

(a) If \( A_1 \cap A_2 \not\prec A_1 \not\prec A_2 \), then \( A_1 \not\prec A_1 \cup A_2 \).

(b) If \( A_1 \not\prec A_2 \not\prec A_1 \cap A_2 \), then \( A_1 \cup A_2 \not\prec A_2 \).

11. I would like to thank a referee for this argument.
Recall the interpretation of Weak DSB, which says that the decision-maker cannot be made worse off by a chance that the lottery that she will consume is chosen from a “good” set instead of a “bad” set. The interpretation of Two-Set Union is analogous and even more appealing. If $\emptyset \neq A_1 \cap A_2 \not\sim A_1 \not\sim A_2$, the decision-maker should prefer $A_1 \cup A_2$ even more compared with the union of disjoint sets $A_1$ and $A_2$ because not only is there a chance that the lottery that she will consume is chosen from a “good” set instead of a “bad” set but also “bad lotteries” (elements of $A_1 \cap A_2$) of the “good set” belong to the “bad” set.

4.1. Representation theorem, measure of attitude to ambiguity

**Theorem 3.** If a preference $\preceq$ defined on the family of all polyhedra $\mathcal{V}$ satisfies Set $S$-Solvability, Set $S$-Independence, GDSB, and Two-Set Union, then there exists a unique $\alpha \in (0, 1)$ such that

$$\preceq \sim H_{\alpha},$$

on the subfamily of convex polyhedra.

**Proof.** See Appendix A.

To prove Theorem 3, one can show (in a manner similar to Theorem 1) that there exists $\alpha$ such that for any pair of lotteries $l_{\min} \not\sim l_{\max}$,

$$\text{Co}(l_{\min}, l_{\max}) \sim \alpha l_{\max} + (1 - \alpha)l_{\min};$$

therefore, it remains to show that

$$A \sim \text{Co}(l_{\min}, l_{\max}), \quad (3)$$

for every convex polyhedron $A$, where $l_{\max} = \arg\max_{l \in A} U(l)$ and $l_{\min} = \arg\min_{l \in A} U(l)$.

To see the argument, suppose that $\dim A = 2$. Consider first the triangle $A = \text{Co}(l_{\min}, l, l_{\max})$ shown in Figure 2(a). Take a lottery $l' \in \text{Co}(l_{\min}, l_{\max})$ such that $\text{Co}(l, l') \sim \text{Co}(l_{\min}, l_{\max})$. Note that $B = \text{bd}A$ can be represented as

$$[0, 1][l, l'] + (1 - [0, 1])[l_{\min}, l_{\max}], \quad (4)$$

with the uniqueness property being satisfied. Weak GDSB implies that $[l, l'] \sim \text{Co}(l, l'), [l_{\min}, l_{\max}] \sim \text{Co}(l_{\min}, l_{\max})$, and $B \sim A$. Therefore, together with (4), it also implies that $A \sim \text{Co}(l_{\min}, l_{\max})$.

Consider now a polyhedron $A$ with more (than three) vertices shown in Figure 2(b), and suppose that (3) holds for all polyhedra with fewer vertices than $A$. Represent $A$ as $A_1 \cup A_2$ as in Figure 2(b). Then, $A_1, A_2$, and $A_1 \cap A_2$ are polyhedra with fewer vertices than $A$, and, therefore, $A_1, A_2, A_1 \cap A_2 \sim \text{Co}(l_{\min}, l_{\max})$. Apply Two-Set Union and the inductive assumption.\(^\text{\scriptsize 12}\)

Corollary 2 generalizes to the preferences that satisfy the axioms from Theorem 3, but only if it is required in condition (2) that the sets $A$ and $B$ are convex polyhedra. More precisely, suppose that $\preceq_1$ and $\preceq_2$ defined on $\mathcal{V}$ satisfy Set $S$-Solvability, Set $S$-Independence, GDSB, and Two-Set Union. Then condition (2) is satisfied for any pair of convex polyhedra $A$ and $B$ such that $A \subset B$ and $B$ is congruent if and only if $a_1 \leq a_2$. I show in Example 4 that convex polyhedra cannot be replaced here with arbitrary polyhedra.

\(^{12}\) Note that the presented argument requires some vertices of $A$ to lie to the right of the segment $\text{Co}(l_{\min}, l_{\max})$ and others to lie to the left of $\text{Co}(l_{\min}, l_{\max})$. The opposite, more complicated, case is omitted in this sketch of the proof.
Remark 3. It follows from the proof of Theorem 3 that Weak GDSB is required only for $P = \{p\}$ for some $p \in [0, 1]$ or $P = \{0, 1\}$ or $[0, 1]$. Two-Set Union is required only when $A_1 \sim A_2 \sim A_1 \cap A_2$. Strict GDSB is required only when one of the sets $A_1$, $A_2$ consists of a single lottery and $P = [0, 1]$, or both sets $A_1$, $A_2$ consist of a single lottery and $P = \{0, 1\}$. It also follows from the proofs that Set S-Independence is required only when either $A_1$ or $A_2$ consists of a single lottery.

Some of the weaker axioms may actually be more appealing; for instance, Strict GDSB seems slightly less appealing for certain sets $P$. To see this, let $A_1$ be a continuum of lotteries $[0, 1] \{1\} + (1 - [0, 1]) \{0, 1\}$, let $A_2$ be the single lottery that yields $0$ for sure, and

© 2007 The Review of Economic Studies Limited
let $P = [0, 1]$. The preference with $A_1 \sim A_1 \cup A_2$ does not seem unreasonable because the decision-maker may (subjectively) find it impossible that Nature will pick the single lottery $\emptyset$ from the continuum of lotteries $A_1 \cup A_2$. No argument of this sort applies to $P = [0, 1]$.

It is possible to prove a counterpart of Theorems 1 and 3 when dispensing with Set S-Independence. More precisely, it can be shown that for any preference that satisfies the other axioms imposed in Theorem 1 (respectively, Theorem 3), the ranking a closed set (respectively, convex polyhedron) coincides with the ranking of its two-element subset consisting of the maximal and minimal element (respectively, the segment joining the maximal and minimal elements). Moreover, it can be shown that, in terms of Figure 1, the indifference curves are downward-sloping (not necessarily parallel), straight lines joining the sides $Co((0, 0), (0, 1))$ and $Co((0, 1), (1, 1))$ with $Co((0, 0), (1, 1))$.

One can then provide an analytical representation of preferences with such indifference curves, similar in style to Dekel (1986). Indeed, the preferences that satisfy the axioms of Theorems 1 and 3 but not Set S-Independence generalize $\preceq^{H_{u,a}}$ in a similar manner to how the preferences over single lotteries that satisfy Betweenness generalize the preferences that satisfy Independence; the indifference curves, in $\Delta$, of preferences that satisfy Solvability and Independence are parallel hyperplanes, and if Independence gets replaced with Betweenness, the indifference curves are still hyperplanes but they need not be parallel. Dekel (1986) generalizes the expected utility and obtains a representation similar in form to the expected utility, in which the utilities of lotteries are defined implicitly.

A potential extension of this idea would be to characterize axiomatically preferences over sets of lotteries such that the ranking of a set is determined by the maximal and minimal element, and the indifference curves, represented as points of the triangle $Co((0, 0), (0, 1), (1, 1))$, are not necessarily linear but differentiable (or locally linear). One would presumably obtain a representation in the style of Machina (1982) instead of Dekel (1986).

4.2. Violation of Set Continuity, tightness of axioms
I first exhibit two examples of preferences that violate Set Continuity and satisfy the other axioms imposed in Theorem 3. Next, I establish the mutual independence of the axioms from Theorem 3.

Example 4. Let $n = 2$. Given a polyhedron $A$, let $A_1, \ldots, A_m$ denote its components, that is, $A = A_1 \cup \ldots \cup A_m$, where $A_1, \ldots, A_m$ are pairwise disjoint and connected polyhedra, that is, points or segments. Now, given a utility index $u : X \to R$, define

$$G_{u,1/2}(A) = \frac{1}{m} \sum_{i=1}^{m} H_{u,1/2}(A_i)$$

and

$$A \preceq^{G_{u,1/2}} B \text{ if and only if } G_{u,1/2}(A) \leq G_{u,1/2}(B).$$

Obviously, $G_{u,1/2}$ coincides with $H_{u,1/2}$ given by (1) for $a = 1/2$ on the subfamily of convex polyhedra. It is easy to check that $A_1 \preceq^{G_{u,1/2}} A_2 \preceq^{G_{u,1/2}} A_3 \preceq^{G_{u,1/2}} \ldots$ for the sequence of sets $A_1, A_2, A_3, \ldots$ described at the beginning of this section; this implies that $\preceq^{G_{u,1/2}}$ violates Set Continuity. In Appendix B, I show that $\preceq^{G_{u,1/2}}$ satisfies Set S-Solvability, Set S-Independence, and GDSB.

The present example also exhibits preferences that satisfy GDSB, but that would violate even DSB if I did not assume that the triple $A_1, A_2, P$ has to satisfy the uniqueness property. Indeed, suppose that $X = [0, 1]$, $u(0) = 0$, and $u(1) = 1$; let $A_1 = \{x_1, x_2\}$ and $A_2 = [l, x_2]$, where $x_1$ and
Then $G_{u,1/2}(A_1) = 0.5$, $G_{u,1/2}(A_2) = 0.6$, and $G_{u,1/2}(A_1 \cup A_2) = 0.4$, which means that $A_1 \cup A_2 \prec A_1 \prec A_2$.

Finally, compare the decision rules $G_{u,1/2}$ and $H_{u,2/5}$. The decision-maker with the latter preference is more objective ambiguity averse on the family of convex polyhedra. However, it is possible that she prefers a congruent set $B$ to a set $A \subset B$, and the decision-maker with the former preference prefers $A$ to $B$; for instance, this happens for $A = A_1$ and $B = A_1 \cup A_2$.

**Example 5.** Given a polyhedron $A \in \mathcal{V}$, let $\dim A$ stand for the dimension of $A$. Let 

$$F(A) = A - \bigcup \{U : U \text{ is an open subset of } \Delta \text{ and } \dim U \cap A < \dim A\}.$$ 

That is, $F(A)$ obtains from $A$ by removing the interiors (in the space $\Delta$) of all simplices $B$ with $\dim B < \dim A$, or in other words, by taking the union of all simplices $C \subset A$ with $\dim C = \dim A$. Figure 3 provides an example. Given a utility index $u : X \to \mathbb{R}$ and $\alpha \in (0, 1)$, define

$$F_{u,\alpha}(A) = \alpha \left( \min_{l \in F(A)} U(l) \right) + (1 - \alpha) \left( \max_{l \in F(A)} U(l) \right)$$

and

$$A \preceq F_{u,\alpha} B \text{ if and only if } F_{u,\alpha}(A) \leq F_{u,\alpha}(B).$$

In the interpretation, the decision-maker with the preference $\preceq F_{u,\alpha}$ believes that Nature chooses the lottery that she will consume from the highest dimensional simplices, so she disregards lower dimensional simplices when she chooses between polyhedra.

The preference $\preceq F_{u,\alpha}$ satisfies GDSB, as does the preference $\preceq H_{u,\alpha}$, and

$$F(P A_1 + (1 - P) A_2) = F(P) F(A_1) + (1 - F(P)) F(A_2).$$

Set $S$-Independence and Set $S$-Solvability are straightforward to check. On the other hand, $\preceq F_{u,\alpha}$ violates Set Continuity; for example, if $A_n$ consists of a “bad” lottery and a full-dimensional
simplex in the \((1/n)\)-ball around a “good” lottery, then \(A_n\) converges in the Hausdorff metric to the two-element set \(A\) consisting of the “bad” lottery and the “good” lottery, but \(F_{u,\alpha}(A_n)\) does not converge to \(F_{u,\alpha}(A)\).

**Example 6.** Consider the uniform probability distribution over a simplex \(\Delta\) with \(\dim \Delta \geq 2\) and the family of all (not necessarily convex) polyhedra. Represent a polyhedron \(A\) as the union of the set of simplices whose interiors (in \(A\)) are non-empty and pairwise disjoint. Let \(m_S\) be the \(k\)-dimensional Lebesgue measure on the hyperplane containing simplex \(S\), where \(k = \dim S\).

Given a utility index \(u\), define

\[
M_u(A) = \frac{\sum_{i=1}^{k} (\int_{S_i} U(l)dm_{S_i})}{\sum_{i=1}^{k} m_{S_i}(S_i)}
\]

and

\[A \prec^M_u B\] if and only if \(M_u(A) \leq M_u(B)\).

Note that, despite the multiplicity of representations \(A = S_1 \cup \cdots \cup S_k\) with the required property, \(M_u(A)\) is uniquely determined. In the interpretation, \(M_u(A)\) is the mass of the centre of gravity of the set \(A\), where the mass of each lottery from the set \(A\) is equal to its expected utility.

It is easy to see that \(\prec^M_u\) satisfies Set S-Solvability and Set S-Independence because the centre of gravity of the set \(pA + (1 - p)\{l\}\) coincides with \(pg + (1 - p)l\), where \(g\) stands for the centre of gravity of the set \(A\). The preference \(\prec^M_u\) also satisfies DSB and Two-Set Union, but violates GDSB.

To see that DSB is satisfied, define \(\int_A U = \sum_{i=1}^{k} (\int_{S_i} U(l)dm_{S_i})\) and \(m_A = \sum_{i=1}^{k} m_{S_i}(S_i)\) and notice that

\[
M_u(A_1 \cup A_2) = \frac{(\int_{A_1 \cup A_2} U)}{m_{A_1 \cup A_2}} = \frac{\int_{A_1} U + \int_{A_2} U}{m_{A_1 \cup A_2}} = \frac{m_{A_1} \int_{A_1} U}{m_{A_1 \cup A_2} m_{A_1}} + \frac{m_{A_2} \int_{A_2} U}{m_{A_1 \cup A_2} m_{A_2}} = \frac{m_{A_1}}{m_{A_1 \cup A_2}} M_u(A_1) + \frac{m_{A_2}}{m_{A_1 \cup A_2}} M_u(A_2).
\]

The argument that \(\prec^M_u\) satisfies Two-Set Union is analogous and the details are omitted.

Figure 4 provides an argument that GDSB is violated. It suggests that decision rules that consider the volume of lotteries with certain utility typically violate GDSB.

It is easy to see that the preferences from Example 1 violate Set S-Independence and satisfy the other axioms imposed in Theorem 3 and that the preferences from Example 2 violate Set S-Solvability and satisfy the other axioms imposed in Theorem 3.

Finally, it follows from the proof that Theorem 3 for simplices can be proven without Two-Set Union. However, it turns out that the power of GDSB is rather limited for polyhedra other than simplices because the operation \(PA + (1 - P)B\) (with the uniqueness property) applies to a relatively small number of polyhedra; thus, little can be said beyond simplices without Two-Set Union (or, possibly, other axioms). There exists a preference \(\prec\) defined on the family of all polyhedra \(\mathcal{V}\) such that \(\prec\) satisfies S-Solvability, S-Independence, and GDSB, \(\prec\) coincides with \(\prec^H_u,\alpha\) on all simplices, yet \(\prec\) differs from \(\prec^H_u,\alpha\) on some convex polyhedra.\(^{13}\)

\(^{13}\) More precisely, consider the simplex \(\Delta\) with \(\dim \Delta = 2\) and the family of all convex polyhedra contained in \(\Delta\). For every member of this family \(A\), let

\[
v_{\min} = \arg \min_{l \in A} U(l) \quad \text{and} \quad v_{\max} = \arg \max_{l \in A} U(l).
\]
5. RELATED LITERATURE

In this section, I provide more details regarding the relation between my axioms and results and those from the existing literature on decision-making under complete uncertainty, as well as on the relation between this paper and the paper of Ahn (2003), which studies the same model.

Papers on decision-making under complete uncertainty typically impose axioms similar in style to Set S-Independence and DSB. A cornerstone paper of this literature, Kannai and Peleg (1984), assumes the Gärdenfors principle:14 \{x\} \succ \{y\} implies \{x\} \succ \{x, y\} \succ \{y\}, that is, Strict DSB in the particular case in which \(A_1\) and \(A_2\) are both singletons. It also assumes the following independence axiom: if \(A_1 \prec A_2\), and \(x \notin A_1 \cup A_2\), then \(A_1 \cup \{x\} \prec A_2 \cup \{x\}\), which can be restated as \(A_1 \preceq A_2\) implies \(P A_1 + (1 - P)A \preceq P A_2 + (1 - P)A\), where \(P = [0, 1]\) and \(A = \{x\}\) is a singleton, with \(x \notin A_1 \cup A_2\) corresponding to the uniqueness property from the present setting.

A typical result from the literature on complete uncertainty says that, under the two axioms quoted above (or some related axioms), the preferences are determined by a rule that involves only the maximal and minimal elements of a set. This sometimes leads, depending on the axioms, to an impossibility result (as in Kannai and Peleg) or to a characterization of some particular rules (see, for instance, Bossert et al., 2000). However, these rules must differ from rule (1) because

If the polyhedron \(A\) has the following two properties

(a) \(U(v) \geq (1/2)U(v_{\max}) + (1/2)U(v_{\min})\) for all vertices \(v \neq v_{\min}\),
(b) at least one vertex of \(A\) lies to the right of the segment \(C(v_{\min}, v_{\max})\), and at least one vertex of \(A\) lies to the left of the segment \(C(v_{\min}, v_{\max})\),

then let \(v_{\text{right}}^*\) and \(v_{\text{left}}^*\) be the vertices that yield the highest expected utility among those to the right and those to the left, respectively.

Then let

\[
H(A) = (1/2)H_{u,1/2}(A) + (1/2)\min\{U(v_{\text{right}}^*), U(v_{\text{left}}^*)\}.
\]

Otherwise, if the polyhedron \(A\) does not have properties (a) and (b), let \(H(A) = H_{u,1/2}(A)\).

It transpires that this rule \(H\) extends to the family of all (not necessarily convex) polyhedra in such a way that the three axioms imposed in Theorem 3 are satisfied. However, the construction is rather complicated and not significantly instructive, so the details are omitted.

14. The name of the principle recognizes the use of this axiom by Gärdenfors (1976).
the literature on decision-making under complete uncertainty assumes only ordinal properties of preferences over single objects.\textsuperscript{15}

The axioms postulated in this paper are more powerful than those in decision-making under complete uncertainty in the sense that they exploit the additional structure of lotteries; in addition, the postulated betweenness axioms are stronger (in the sense that they imply, but are not implied by) than the Gärdenfors principle, even abstracting from the lottery structure. On the other hand, I assume in Set S-Independence that \( P \) is a singleton and Kannai and Peleg assume, in their independence, that \( P = \{0, 1\} \). I argue that this is an important difference; their axiom seems to be strong and it would probably be violated by many decision-makers.\textsuperscript{16}

To see this, let \( A_1 = \{\$1550, \$1450\}, A_2 = \{\$1500\}, \) and \( x = \$0 \). It seems plausible to strictly prefer \( A_2 \) to \( A_1 \) (because of objective ambiguity aversion) and, simultaneously, to strictly prefer \( A_1 \cup \{x\} = \{\$1550, \$1450, \$0\} \) to \( A_2 \cup \{x\} = \{\$1500, \$0\} \). The intuition here is that, although a decision-maker is always made worse off by adding a “bad” lottery to a set, adding such a “bad” lottery makes much more of a difference when the initial set consists of just one “good” lottery than when it consists of two “good” lotteries.

Ahn (2003) studies exactly the same model as is presented here. However, he restricts attention to the sets that are either singletons or closures of their own interiors. He characterizes the rule

\[
H(A) = \frac{\int_{A} \phi \circ UD \mu}{\mu(A)},
\]

where \( U \) is the expected utility, \( \phi \) is an increasing and continuous real-valued function, and \( \mu \) is a non-atomic probability measure on \( \Delta(X) \). He interprets \( \phi \) and \( \mu \) as measures of attitude to (what I call) objective ambiguity.

Ahn assumes Set Continuity, a stronger version of DSB (\( i.e. \) that \( A \preceq B \) implies \( A \preceq A \cup B \preceq B \) and \( A \prec B \) implies \( A \prec A \cup B \prec B \)), as well as two other axioms: Balancedness and Divisibility. Balancedness says that if \( [A \cup B] \cap [C \cup D] = \emptyset \) and \( A \sim B \succ C, D \) (or \( A \sim B \prec C, D \)), then \( A \cup C \succeq B \cup C \) implies \( A \cup D \succeq B \cup D \); Divisibility is a technical axiom that concerns the sets that are closures of their own interiors. The \( \alpha \)-maxmin rule satisfies Balancedness, but violates the strict part of Ahn’s version of betweenness. Divisibility makes little sense for sets, such as those considered in this paper, that are not closures of their own interiors.

The decision rule characterized by Ahn does satisfy Two-Set Union and typically violates GDSB and Set S-Independence. However, an important particular case, where \( \phi \) is a linear function and \( \mu \) is the Lebesgue measure, satisfies also Set S-Independence (see Example 6). That being so, one may wonder why Ahn’s representation is not one of the decision rules characterized in Theorem 1. To see why it is not included, note that Ahn’s representation makes sense only for a narrow family of sets, which (in particular) have the same dimension as the simplex \( \Delta \), while Theorem 1 characterizes preferences defined on the family of all closed subsets of \( \Delta \). It is this larger domain of preferences that precludes Ahn’s representation of particular \( \phi \) and \( \mu \) from being among the decision rules characterized in Theorem 1.

APPENDIX A

\textit{Proof of Theorems 1 and 3.} I will prove Theorems 1 and 3 by a sequence of lemmas. I will denote by \( l_\alpha \) the lottery that yields outcome \( x_n \) with probability \( \alpha \) and outcome \( x_1 \) with probability \( 1 - \alpha \).

\textbf{Lemma 4.} Let \( \preceq \) satisfy Set S-Independence and Set S-Solvability.

15. The preference that resembles most closely the \( \alpha \)-maxmin rule is probably the rule that identifies sets with their median elements, characterized (using a rather different set of axioms) by Nitzan and Pattanaik (1984).

16. A similar criticism applies to other sorts of independence assumed in the literature on complete uncertainty, for example, intermediate independence from Barberà \textit{et al.} or extension independence from Bossert \textit{et al.}
(i) If $\preceq$ satisfies Strict DSB, then there exists $a \in (0, 1)$ such that, for any $0 \leq a < b \leq 1$,

$$[l_a, l_b] \sim l_{ab + (1-a)a}. \quad (5)$$

(ii) If $\preceq$ satisfies Strict GDSB, then there exists $a \in (0, 1)$ such that, for any $0 \leq a < b \leq 1$,

$$[a, b]x_n + (1 - [a, b])x_1 \sim l_{ab + (1-a)a}. \quad (6)$$

**Proof.** Since $x_1 \prec x_n$, the assumption that the decision-maker evaluates single lotteries by their expected utility implies that the ranking of the lottery $l_a$ strictly increases with $a$.

Strict DSB implies that

$$x_1 \prec [x_1, x_n] \prec x_n$$

and Strict GDSB implies that

$$x_1 \prec [0, 1]x_n + (1 - [0, 1])x_1 \prec x_n;$$

in the former case, apply Strict DSB to $A_1 = \{x_1\}$ and $A_2 = [x_n]$ and in the latter case, apply Strict GDSB to the same sets $A_1$ and $A_2$ and $P = [0, 1]$.

Thus, by Set S-Solvability, there exists $a \in (0, 1)$ such that

$$[x_1, x_n] \sim l_a$$

and

$$[0, 1]x_n + (1 - [0, 1])x_1 \sim l_a,$$

respectively.

Referring to Figure 1, Set S-Independence implies that the decision-maker is indifferent between all points from the line joining $(0, 1)$ and $(a, a)$; indeed, apply Set S-Independence to $A_1 = \{x_1, x_n\}$ or $[0, 1]x_n + (1 - [0, 1])x_1$, $A_2 = A = \{l_a\}$, and $p$ moving along the segment $[0, 1]$. Furthermore, applying Set S-Independence again, it may be derived that the indifference curves coincide with straight lines parallel to the line joining $(0, 1)$ and $(a, a)$; indeed, apply Set S-Independence to $A_1 = \{l_a\}$. $A_2$ represented by a point moving along the line joining $(0, 1)$ and $(a, a)$, $A = \{x_1\}$ or $A = \{x_n\}$, and $p$ moving along the segment $[0, 1]$.

The conditions (5) and (6) express algebraically the statement that the indifference curves of $\preceq$ are straight and parallel downward-sloping lines.

More generally, Lemma 4 holds when $x_1$ and $x_n$ are replaced with any pair of lotteries. However, $\alpha$ is potentially different for different pairs of lotteries.

**Lemma 5.** Suppose that $\preceq$ satisfies Set S-Solvability and Weak GDSB. If $A$ is a simplex and $B = \text{bd}A$, then $A \sim B$.

**Proof.** I shall first show, by induction with respect to the dimension of $A$, that there exist lotteries $l_1, l_2 \in B$ such that

$$l_1 \preceq B \preceq l_2.$$  Suppose that $A$ is a segment. Then $B = \{l_1, l_2\}$; assume, without loss of generality, that $l_1 \preceq l_2$. Since $B = [0, 1][l_2] + (1 - [0, 1])[l_1]$, $l_1 \preceq B \preceq l_2$ by Weak GDSB. Suppose that $A$ is an $n$-dimensional simplex; let $v_1, \ldots, v_{n+1}$ denote the vertices of $A$; without loss of generality, assume that $v_1 \preceq \cdots \preceq v_{n+1}$. Take a lottery $l \in \text{intCo}(v_1, \ldots, v_n)$.

Then $B = [0, 1][\text{bdCo}(v_1, \ldots, v_n)] + (1 - [0, 1])[v_{n+1}, l]$. Suppose that $[v_{n+1}, l] \preceq \text{bdCo}(v_1, \ldots, v_n)$; the opposite case is analogous. By Weak GDSB, $[v_{n+1}, l] \preceq B \preceq \text{bdCo}(v_1, \ldots, v_n)$ and by the inductive assumption, $l_1 \preceq [v_{n+1}, l]$ for some $l_1 \in [v_{n+1}, l]$ and $\text{bdCo}(v_1, \ldots, v_n) \preceq l_2$ for some $l_2 \in \text{bdCo}(v_1, \ldots, v_n)$.

If $l_1 \prec B \prec l_2$ or $l_1 \sim l_2$, then there exists a lottery $l$ from the interior of $A$ such that $l \sim B$, which follows from Set S-Solvability in the former case and it is straightforward in the latter case. Since $A = [0, 1][l + (1 - [0, 1])B]$, $A \sim B$ by Weak GDSB. Suppose, therefore, that $l_1 \prec B \prec l_2$; the case $l_1 \sim B \prec l_2$ is analogous. Then take a lottery $l$ belonging to the interior of $A$ such that $l \sim (1 - \varepsilon)[l_2] + \varepsilon l_1$. By the previous argument, $l \prec A \not\sim B$. However, this holds for every $\varepsilon > 0$, so $A \sim l_2 \sim B$.  

**Lemma 6.** Let $\preceq$ satisfy Set S-Independence.

(i) If $\preceq$ satisfies Set Continuity and DSB, then for any $l_1 \sim l_3$ and $l_2 \sim l_4$,

$$[l_1, l_2] \sim [l_3, l_4]. \quad (7)$$

(ii) If $\preceq$ satisfies Set S-Solvability and GDSB, then for any $l_1 \sim l_3$ and $l_2 \sim l_4$,

$$[0, 1][l_2] + (1 - [0, 1])l_1 \sim [0, 1][l_4] + (1 - [0, 1])l_2. \quad (8)$$

© 2007 The Review of Economic Studies Limited
Lemma 4), there exists an apply Set S-Independence. Then such that $p \sim [0, 1]$. Further, without loss of generality, it can be assumed that $[0, 1]l + (1 - [0, 1])l_3$ are parallel, the interior of (at least) one of them lies in the interior of $\Delta$: say it is the interior of $[0, 1]l_4 + (1 - [0, 1])l_3$. Now take a lottery $l_5 \sim \{l_1, l_2\}$, such that the straight lines passing through $l_5$ and $l_1$ and passing through $l_5$ and $l_2$ intersect the interior of $[0, 1]l_4 + (1 - [0, 1])l_3$; let $l_6$ and $l_7$ stand for the intersection points (see Figure A.1(a)). Observe that $\{l_6, l_7\} \sim \{l_1, l_2\}$;

indeed, note that $\{l_6, l_7\} = p\{l_1, l_2\} + (1 - p)\{l_5\}$ for some $p \in (0, 1)$ (and obviously $\{l_5\} = p\{l_5\} + (1 - p)\{l_5\}$), and apply Set S-Independence.

Take $l_8 \in [0, 1]l_4 + (1 - [0, 1])l_3$ such that $l_8 \sim l_5$.

Then $\{l_6, l_7\} = p\{l_3, l_4\} + (1 - p)\{l_8\}$. This yields that $\{l_6, l_7\} \sim \{l_3, l_4\}$.

Indeed, $\{l_6, l_7\} \sim l_8$, and it follows immediately from Lemma 4 that (1) if $\{l_3, l_4\} \prec l_8$, then $p\{l_3, l_4\} + (1 - p)\{l_8\} \prec l_8$ for every $p \in (0, 1)$, (2) if $l_8 \prec \{l_3, l_4\}$, then $l_8 \prec p\{l_3, l_4\} + (1 - p)\{l_8\}$ for every $p \in (0, 1)$.

A similar argument yields (8) for parallel segments $[0, 1]l_2 + (1 - [0, 1])l_1$ and $[0, 1]l_4 + (1 - [0, 1])l_3$.

I shall now show that (7) holds for every pair of non-parallel $[0, 1]l_2 + (1 - [0, 1])l_1$ and $[0, 1]l_4 + (1 - [0, 1])l_3$. Suppose that $\{l_1, l_2\} \prec \{l_3, l_4\}$. Therefore, by Lemma 4 (see the remark following the proof of Lemma 4), there exists an $l \prec l_4$ belonging to $[0, 1]l_4 + (1 - [0, 1])l_3$ such that

$\{l_1, l_2\} \sim \{l_3, l_4\}$.

Without loss of generality, it can be assumed that $l_3$ and $l_1$ coincide; indeed, $[0, 1]l_4 + (1 - [0, 1])l_3$ can be replaced with a parallel segment, which has the required property. Consider the set $\{l_1, l_2, l_3\}$. On the one hand, $\{l_1, l_2\} \prec \{l_1, l_2, l_3\}$ by Strong DSBB, as $\{l_1, l_2, l_3\} = \{l_1, l_2, l_3\}$; on the other, $\{l_1, l_2\} \sim \{l_1, l_2, l_3\}$ by Weak DSBB and Set Continuity, as $\{l_1, l_2, l_3\}$ is the limit (in the Hausdorff metric) of $\{l_1, l_2\} \cup \{l_3, l_3\}$, where $[0, 1]l_3 + (1 - [0, 1])l_3$ is a sequence of segments parallel to $[0, 1]l_3 + (1 - [0, 1])l_3$ that converges (in the Hausdorff metric) to $[0, 1]l_3 + (1 - [0, 1])l_3$.

Finally, I have to show (8) for every pair of non-parallel $[0, 1]l_2 + (1 - [0, 1])l_1$ and $[0, 1]l_4 + (1 - [0, 1])l_3$. Suppose that $[0, 1]l_2 + (1 - [0, 1])l_3 \prec [0, 1]l_4 + (1 - [0, 1])l_3$. Therefore, by Lemma 4 (see the remark following the proof of Lemma 4), there exists an $l_6 \prec l_4$ belonging to $[0, 1]l_4 + (1 - [0, 1])l_3$ and $l_5 \succ l_1$ belonging to $[0, 1]l_2 + (1 - [0, 1])l_1$ such that

$[0, 1]l_2 + (1 - [0, 1])l_5 \sim [0, 1]l_6 + (1 - [0, 1])l_3$.

Further, without loss of generality, it can be assumed that $l_6$ belongs to the interior of $[0, 1]l_2 + (1 - [0, 1])l_1$; indeed, $[0, 1]l_6 + (1 - [0, 1])l_3$ can be replaced with a parallel segment, which has the required property. See Figure A.1(b).

Notice that triangle $\text{Co}((l_5, l_5, l_3))$ can be represented as

$[0, 1]\text{Co}((l_5, l_2)) + (1 - [0, 1])\{l_3\}$.

Since $l_3 \prec l$ for every $l \in \text{Co}((l_5, l_2))$, $\text{Co}((l_5, l_5, l_3)) \prec \text{Co}((l_5, l_2))$ by Strong DSBB (it is straightforward to check the uniqueness property required in Strong DSBB). Notice next that the boundary of the same triangle, that is, $B = \text{Co}((l_5, l_2)) \cup \text{Co}((l_2, l_3)) \cup \text{Co}((l_3, l_3))$, can be represented as

$[0, 1]l_2 + (1 - [0, 1])\{l_5, l_3\}$,

(again, it is straightforward to check the uniqueness property). By Weak DSBB and Lemma 5, $\text{Co}((l_5, l_2, l_3)) \sim B \sim l_5, l_2 \sim \text{Co}((l_5, l_5))$, a contradiction. ||
Proof of Theorem 1. Given a closed subset $A \subset \Delta$, let $l_{\min}, l_{\max} \in A$ be such that $U(l_{\min}) \leq U(l) \leq U(l_{\max})$ for all other $l \in A$. Assume that $l_{\min} \prec l_{\max}$; otherwise the result obtains immediately by the iterative application of Weak DSB and next Set Continuity. Observe first that, for every $\varepsilon > 0$, there exist pairwise disjoint sets $A_i = \{a_i, b_i, c_i\}$, where $a_i \prec b_i \prec c_i$ for $i = 1, \ldots, k$, such that $\text{dist}(A, \bigcup_{i=1}^k A_i) < \varepsilon$ and $\text{dist}(\{l_{\min}, l_{\max}\}, \{a_i, c_i\}) < \varepsilon$. Indeed, take a finite set $B = \{b_1, \ldots, b_k\}$ such that $\text{dist}(A, B) < \varepsilon$ and $l_{\min} \prec b_i \prec l_{\max}$ ($i = 1, \ldots, k$). Then take $a_1, \ldots, a_k$ from the $\varepsilon$-ball around $l_{\min}$ and take $c_1, \ldots, c_k$ from the $\varepsilon$-ball around $l_{\max}$. The sets $A_i = \{a_i, b_i, c_i\}, i = 1, \ldots, k$ have the required properties.

Assume that $a_i \prec b_i \prec c_i$; the assumption that the preference is strict is without loss of generality due to Set Continuity. I shall now show that

$$A_i \sim \{a_i, c_i\}.$$ (9)
Suppose that \( b_j \preceq \{a_i, c_j\} \); the opposite case is analogous. By Weak DSB applied to \( \{b_j\} \) and \( \{a_i, c_j\} \), \( A \preceq \{a_i, c_j\} \). Take a lottery \( l \) from a \( \delta \)-ball around \( c_j \); if \( \delta \) is small enough, then \( \{a_i, c_j\} \prec \{b_j, l\} \) by Lemmas 4 and 6. Thus, \( \{a_i, c_j\} \preceq \{a_i, b_j, c_j, l\} \); indeed, without loss of generality assume that \( l \notin \{a_i, c_j\} \) and apply Weak DSB to \( \{a_i, c_j\} \) and \( \{b_j, l\} \). Note that \( \{a_i, b_j, c_j, l\} \) converges to \( A \) in the Hausdorff metric as \( \delta \to 0 \). This yields \( \{a_i, c_j\} \preceq A \).

I shall show now that \( A \sim \{l_{\min}, l_{\max}\} \), which will end the proof, by virtue of Lemmas 4 and 6. Take \( A_{\min}, A_{\max} \in \{A_1, \ldots, A_k\} \) such that \( A_{\min} \preceq A_i \preceq A_{\max} \) for all \( i \). By applying Weak DSB iteratively, \( A_{\min} \preceq \bigcup_{i=1}^{k} A_k \preceq A_{\max} \). However, \( \bigcup_{i=1}^{k} A_k \) converges to \( A \) in the Hausdorff metric as \( \varepsilon \to 0 \), \( A_{\min} \sim \{l_{\min}, l_{\max}\} \), \( A_{\max} \sim \{l_{\max}, l_{\max}\} \) and both \( \{a_{\min}, c_{\min}\} \) and \( \{a_{\max}, c_{\max}\} \) converge to \( \{l_{\min}, l_{\max}\} \) in the Hausdorff metric as \( \varepsilon \to 0 \).

Theorem 3 for segments, that is, simplices (or convex polyhedra) of dimension 1, follows from Lemmas 4 and 6. Lemma 7 establishes Theorem 3 for simplices of higher dimensions.

**Lemma 7.** Suppose \( \preceq \) satisfies Set S-Independence, Set S-Solvability, and GDSB. Then, for every simplex \( A = \text{Co}(l_1, \ldots, l_m) \), where \( l_1 \preceq \cdots \preceq l_m \),

\[
A \sim [0, 1]_m + (1 - [0, 1])l_1.
\]

**Proof.** Apply induction with respect to the number of vertices. Suppose that (10) holds whenever \( A \) has less than \( m \geq 2 \) vertices, and consider \( A \) with \( m \) vertices. If there are \( i, j \in \{2, \ldots, m - 1\} \) such that \( l_i \preceq [0, 1]_m + (1 - [0, 1])l_1 \preceq l_j \), then (10) follows from Weak GDSB and the inductive assumption. Indeed, \( A \) can be represented as

\[
[0, 1] \text{Co}(l_1, \ldots, l_m) - (l_i) + (1 - [0, 1])l_i
\]

where both representations have the uniqueness property. The former representation of \( A \) yields \( A \preceq [0, 1]_m + (1 - [0, 1])l_i \) and the latter representation yields \( [0, 1]_m + (1 - [0, 1])l_i \preceq A \).

Suppose, therefore, that \( l_i \preceq [0, 1]_m + (1 - [0, 1])l_i \) for all \( i = 2, \ldots, m - 1 \); the case of \( [0, 1]_m + (1 - [0, 1])l_1 \) is analogous. Then \( A \preceq [0, 1]_m + (1 - [0, 1])l_i \) by the previous argument and I have only to show that \( [0, 1]_m + (1 - [0, 1])l_1 \preceq A \). There are two cases:

**Case 1** (\( l_i < l_i \) for some \( i = 2, \ldots, m - 1 \)). Take a lottery \( l \in \text{intCo}(l_1, l_{i-1}, l_{i+1}, \ldots, l_m) \) such that

\[
[0, 1]_m + (1 - [0, 1])l_i \preceq [0, 1]_l + (1 - [0, 1])l_i.
\]

**Lemma 5** and **Lemma 6** guarantee that any \( l \) that lies sufficiently close to \( l_m \) has the required property. Notice that \( B = \text{bdCo}(l_1, \ldots, l_m) \) can be represented as

\[
[0, 1]_m + (1 - [0, 1])l_1 + (1 - [0, 1])\text{bdCo}(l_1, l_{i-1}, l_{i+1}, \ldots, l_m),
\]

it is straightforward to see that the representation has the uniqueness property required by Weak GDSB.

By **Lemma 5**, \( \text{bdCo}(l_1, l_{i-1}, l_{i+1}, \ldots, l_m) \sim \text{Co}(l_1, l_{i-1}, l_{i+1}, \ldots, l_m) \) and \( [l, l] \sim [0, 1]_l + (1 - [0, 1])l_i \).

Therefore, by Weak GDSB and the inductive assumption, \( [0, 1]_m + (1 - [0, 1])l_i \preceq [0, 1]_m \preceq B \). Finally, by **Lemma 5**, \( A \sim B \).

**Case 2** (\( l_i \sim l_i \) for \( i = 2, \ldots, m - 1 \)). If also \( l_m \sim l_1 \), then \( A \sim l_1 \). Indeed, represent \( A \) as

\[
[0, 1]_m + (1 - [0, 1])\text{Co}(l_1, \ldots, l_{m-1})
\]

and apply the inductive assumption and Weak GDSB. Suppose, therefore, that \( l_1 \sim l_m \). Then, for every \( \varepsilon \in (0, 1) \), there exists a lottery \( l \in \text{intCo}(l_1, l_3, \ldots, l_m) \) such that

\[
[0, 1]_m + (1 - [0, 1] - \varepsilon)l_1 \preceq [0, 1]_l + (1 - [0, 1])l_2.
\]

An analogous argument to that used in Case 1 yields that \( [0, 1]_m + (1 - [0, 1] - \varepsilon)l_1 \preceq A \). This implies that \( [0, 1]_m + (1 - [0, 1])l_1 \preceq A \). Indeed, if \( A = [0, 1]_m + (1 - [0, 1])l_1 \), then \( [0, 1]_m + (1 - [0, 1])l_1 \preceq A \) for some \( \varepsilon \in (0, 1) \) by Set S-Solvability applied to \( A \). \( A_1 = \{l_1\} \), and \( A_2 = [0, 1]_m + (1 - [0, 1])l_1 \), so \( A = [0, 1]_m + (1 - [0, 1] - \varepsilon)l_1 \) for every \( \varepsilon < \varepsilon^* \).
Proof of Theorem 3. Let $A$ be a convex polyhedron and let $v_{\min}$ and $v_{\max}$ denote vertices of $A$ with the property that $v_{\min} \preceq v \succeq v_{\max}$ for every other vertex $v$. By Lemmas 4 and 6, it suffices to show that

$$A \sim \text{Co}(v_{\min}, v_{\max}).$$

(11)

Apply induction with respect to the dimension of $A$ and then induction with respect to the number of vertices. The hypothesis (11) follows immediately from Lemma 7 when $A$ is a simplex. Since simplices are the only convex polyhedra of dimension 1, it can, without loss of generality, be assumed that the dimension of $A$ exceeds 1.

I can also assume that the segment $\text{Co}(v_{\min}, v_{\max})$ is an edge of $A$. Indeed, suppose that (11) has been shown for all convex polyhedra such that $\text{Co}(v_{\min}, v_{\max})$ is contained in a $k$-dimensional face of $A$, and consider such a polyhedron containing $\text{Co}(v_{\min}, v_{\max})$ in a $(k+1)$-dimensional face $F$, but not containing $\text{Co}(v_{\min}, v_{\max})$ in any $k$-dimensional face. Let $P$ stand for a $k$-dimensional hyperplane containing $v_{\min}, v_{\max}$, and all but two vertices of $F$, such that each side of $P$ contains one of the two remaining vertices. Represent $A$ as the union $A_1 \cup A_2$, where each $A_j$ is the convex hull of $F \cap P$, one of the two vertices of $F = P$, and all vertices of $A - F$. Now apply Two-Set Union and the inductive assumption.

Let $H$ denote the hyperplane consisting of all lotteries $l$ with $l \sim \text{Co}(v_{\min}, v_{\max})$, let $C$ denote the set of all vertices of $A$, and let $C_v$ stand for the set of all neighbour vertices of $v$; when $v = v_{\min}$ or $v_{\max}$, denote $C_v$ by $C_{\min}$ or $C_{\max}$ respectively. The case $v_{\min}, v_{\max} \in H$ is straightforward. Indeed, if $A$ is not a simplex, then there are two non-neighbour vertices $v$ and $v'$ of $A$. Represent $A$ as $\text{Co}(C - \{v\}) \cup \text{Co}(C - \{v'\})$ and apply Two-Set Union and the inductive assumption.

Therefore, suppose that $v_{\min} < \text{Co}(v_{\min}, v_{\max}) < v_{\max}$. There are three cases:

**Case 1** ($C \cap H \neq \emptyset$). Take a $v \in C \cap H$. By the inductive assumption, $B = \text{Co}(C - \{v\}) \sim \text{Co}(v_{\min}, v_{\max})$. Let $B_0 = \text{Co}C_v$. Notice that there exist lotteries $l_1, l_2 \in \text{int}B_0$ such that $l_1 < B < l_2$ and that

$$B = [0, 1]l_1 + (1 - [0, 1])(\text{bd}B - \text{int}B_0),$$

(12)

for $i = 1, 2$, with the uniqueness property being satisfied. Therefore, $\text{bd}B - \text{int}B_0 \sim B$; otherwise, say if $\text{bd}B - \text{int}B_0 \prec B$, (12) for $i = 1$ and Weak GDSB imply $B = B$. Notice that

$$A = [0, 1]v + (1 - [0, 1])(\text{bd}B - \text{int}B_0),$$

with the uniqueness property being satisfied. Therefore, since $v \sim \text{Co}(v_{\min}, v_{\max}) \sim B \sim \text{bd}B - \text{int}B_0$, $A \sim \text{Co}(v_{\min}, v_{\max})$ again by Weak GDSB.

**Case 2** ($C \cap H = \emptyset$, and there exist vertices $v_1 \neq v_{\min}$ with $v_1 \prec A$ and $v_2 \neq v_{\max}$ with $A \prec v_2$). Take vertices $v_1 \neq v_{\min}$ and $v_2 \neq v_{\max}$ such that $v_1 \prec A$ and $v \preceq v_1$ whenever $v$ is a vertex with $v \prec A$ and $A \prec v_2$ and $v_2 \succeq v$ whenever $v$ is a vertex with $v \prec A$.

By the inductive assumption, $B_i = \text{Co}(C - \{v_i\}) \sim \text{Co}(v_{\min}, v_{\max})$, $i = 1, 2$. Let $C_i$ stand for the set of all neighbour vertices of $v_i$, and $D_i = \text{Co}C_i$. Since $C \cap H = \emptyset$, there are $l_1^i, l_2^i \in \text{int}D_i$ such that $l_1^i < B_i < l_2^i$ (see Figure A.2(a)). Notice that

$$B_i = [0, 1]l_1^i + (1 - [0, 1])(\text{bd}B_i - \text{int}D_i),$$

$$j = 1, 2,$$

with the uniqueness property being satisfied. Therefore, by Weak GB, $\text{bd}B_i - \text{int}D_i \sim B_i$.

Since

$$A = [0, 1]v_1 + (1 - [0, 1])(\text{bd}B_1 - \text{int}D_1),$$

with the uniqueness property being satisfied, and $A \preceq (v_{\min}, v_{\max})$ again by Weak GDSB; similarly,

$$A = [0, 1]v_2 + (1 - [0, 1])(\text{bd}B_2 - \text{int}D_2),$$

so $\text{Co}(v_{\min}, v_{\max}) \preceq A$ by Weak GDSB.

**Case 3** ($C \cap H = \emptyset$, and $A \prec v$ for every $v \neq v_{\min}$; the case $v \prec A$ for every $v \neq v_{\max}$ is analogous). By an argument similar to that from Case 2, $\text{Co}(v_{\min}, v_{\max}) \preceq A$. Suppose that $\text{Co}(v_{\min}, v_{\max}) \prec A$. It can, without loss of generality, be assumed that $\text{Co}(v_{\max}, C_{\max} - \{v_{\min}\})$ is a simplex with $\dim \text{Co}(v_{\max}, C_{\max} - \{v_{\min}\}) = \dim A - 1$. Indeed, if $\text{Co}(v_{\max}, C_{\max} - \{v_{\min}\})$ were not a simplex, there would be two non-neighbour vertices $v, v' \in C_{\max} - \{v_{\min}\}$. One then could represent $A$ as $\text{Co}(C - \{v\}) \cup \text{Co}(C - \{v'\})$, and apply Two-Set Union and the inductive assumption.

By the convexity of $A$, $\dim \text{Co}(v_{\max}, C_{\max} - \{v_{\min}\})$ must exceed $\dim A - 2$. If $\dim \text{Co}(v_{\max}, C_{\max} - \{v_{\min}\}) = \dim A$, it is easy to see that either $\text{Co}(v_{\min}, v_{\max})$ could not be an edge of $A$ or $A$ could be represented

© 2007 The Review of Economic Studies Limited
as the union of two convex polyhedra $A_1$ and $A_2$ that satisfy the assumptions of Case 3 such that $v_{\text{min}}, v_{\text{max}} \in A_1 \cap A_2$; by Two-Set Union, if the result holds for $A_1$ and $A_2$, it also holds for $A$.

It can also be assumed that $C \neq \{v_{\text{max}}\} \cup C_{\text{max}}$; otherwise $A = \text{CoC}$ would be a simplex.

Take a vertex $w \in C - \{v_{\text{max}}\} \cup C_{\text{max}}$. There exists a vertex $c \in \{v_{\text{max}}\} \cup C_{\text{max}}$ such that $w$ and $c$ lie on the opposite sides of the hyperplane $P$ containing $\{v_{\text{max}}\} \cup C_{\text{max}} - \{c\}$. If $c \neq v_{\text{max}}$, then represent $A$ as the union of $A_1 = \text{Co}(C - \{c\})$ and $A_2 = \text{Co}(C - \{w\})$. By the inductive assumption and Two-Set Union, $A$ satisfies (11).

Thus, it can, without loss of generality, be assumed that $C - \{v_{\text{max}}\} \cup C_{\text{max}} \cup \{v_{\text{min}}\}$ and $\{v_{\text{max}}\}$ are contained in the opposite sides of the hyperplane $P$ containing $C_{\text{max}} \cup \{v_{\text{min}}\}$.

Let $c_{\text{max}} \in C_{\text{max}}$ have the property that $U(c_{\text{max}}) = \max_{c \in C_{\text{max}}} U(c)$. Consider now the hyperplane $Q$ containing $\{v_{\text{max}}\} \cup [C_{\text{max}} - \{v_{\text{min}}\}]$. Take a vertex $v \in C - \{v_{\text{max}}\} \cup C_{\text{max}} \cup \{v_{\text{min}}\}$ neighbour to some vertex $c \in C_{\text{max}}$, and take a point
\( \nu^* \in \text{Co}(v_{\min}, v) \) such that \( A \sim \text{Co}(v^*, c_{\max}) \) (see Figure A.2(b)); such a point \( v^* \prec_c A \) exists because \( A \prec_c c_{\max} \), so \( A \prec_c \text{Co}(v^*, c_{\max}) \) whenever \( v^* \sim A \) and \( \text{Co}(v_{\min}, c_{\max}) \prec A \). The straight line that contains \( v \) and \( v^* \) intersects \( Q \) at some point \( v' \) with \( v' \prec c_{\max} \prec v_{\max} \) (see again Figure A.2(b)). \(^{17} \) Indeed, since \( A \) is convex, the plane containing \( v_{\min}, v_{\max}, \) and \( v \) intersects \( \text{Co}(v_{\max}) \) at a point \( c \), and since \( v^* \prec v \) and \( c \prec v_{\max} \), the straight line containing \( v \) and \( v^* \) intersects the straight line that contains \( v_{\max} \) and \( c \).

Let \( B \coloneqq \text{Co}(\{v^*, v\} \cup [C = \{v_{\min}, v_{\max}\}]) \). Since \( v^*, v, \) and \( v' \) lie on a straight line, \( B \) has fewer vertices than \( A \). Thus, \( B \sim \text{Co}(v^*, c_{\max}) \) by the inductive assumption. Since \( A \cap B = \text{Co}(v^*) \cup [C = \{v_{\min}, v_{\max}\}] \) has also fewer vertices than \( A \), \( A \cap B \sim \text{Co}(v^*, c_{\max}) \) by the inductive assumption. By Two-Set Union, \( A \cup B \sim A \). To generate a contradiction with \( \text{Co}(v_{\min}, v_{\max}) \prec A \) I shall now show that

\[
A \cup B \sim \text{Co}(v_{\min}, v_{\max})).
\]

Notice that \( A \cup B \) is a convex polyhedron. Since \( v^*, v \in \text{Co}(v_{\min}, v') \), \( A \cup B \) has at most as many vertices as \( A \). If it has fewer vertices (as in Figure A.2(b)), then (13) follows by the inductive assumption. If it has the same number of vertices, then \( \text{Co}(v_{\max}) \cup C_{\max} \cup \{v'\} \subseteq Q \) cannot be a simplex. Thus, \( \text{Co}(v_{\max}) \cup C_{\max} \cup \{v'\} \) can be represented as \( \text{Co}(v_{\max}) \cup F \), where \( E, F \subseteq \{v_{\max}\} \cup C_{\max} \cup \{v'\} \), \( v_{\max} \in E \cap F \), and \( E \) as well as \( F \) have fewer elements than \( \{v_{\max}\} \cup C_{\max} \cup \{v'\} \). This yields

\[
A \cup B = \text{Co}(E \cup \{C = \{v_{\max}\} \cup C_{\max}\}) \cup \text{Co}(F \cup \{C = \{v_{\max}\} \cup C_{\max}\})
\]

and \( \text{Co}(E \cup \{C = \{v_{\max}\} \cup C_{\max}\}) \) as well as \( \text{Co}(F \cup \{C = \{v_{\max}\} \cup C_{\max}\}) \) is a polyhedron with fewer vertices than \( A \). By the inductive assumption,

\[
\text{Co}(E \cup \{C = \{v_{\max}\} \cup C_{\max}\}) \sim \text{Co}(F \cup \{C = \{v_{\max}\} \cup C_{\max}\}) \sim \text{Co}(v_{\min}, v_{\max}),
\]

which yields (13) by Two-Set Union.

APPENDIX B

The relation \( \sim_{G_{n,1/2}} \) satisfies GDSB: Since \( P A_1 + (1 - P) A_2 \) is connected whenever \( A_1, A_2, \) and \( P \) are connected and \( \sim_{G_{n,1/2}} \) satisfies the two axioms, so does \( \sim_{G_{n,1/2}} \) whenever \( A_1, A_2, \) and \( P \) are connected. Consider now arbitrary polyhedra \( A_1, A_2, \) and \( P \). Suppose first that \( A_1, A_2, \) and \( P \) have the uniqueness property. Represent \( A_i \) as the union of its components:

\[
A_i = \bigcup_{k \in K_i} A_i^k;
\]

similarly represent \( P \) as

\[
P = \bigcup_{m \in M} P^m.
\]

\(^{17} \) Without loss of generality, it can be assumed that the intersection point belongs to \( \Delta \). Indeed, \( l \in \text{int} \Delta \) can be taken such that \( l \sim A \) and a small enough \( p > 0 \) such that the intersection point for \( A \) replaced with that for \( (1 - p)l \) belongs to \( \Delta \). By Weak GDSB, \( pA + (1 - p)l \sim A \), so by Lemmas 4 and 6, if \( \text{Co}(v_{\min}, v_{\max}) \prec A \), then \( \text{Co}(p v_{\min} + (1 - p)l, pv_{\max} + (1 - p)l) \prec A + (1 - p)l \).

© 2007 The Review of Economic Studies Limited
It is easy to check that if \( n = 2 \), then
\[
(P_A + (1 - P)A_2 = \bigcup_{k_1 \in K_1, k_2 \in K_2, m \in M} p^{m} A_{1}^{k_1} + (1 - p^{m}) A_{2}^{k_2}
\]
is the representation of \( P_A + (1 - P)A_2 \) as the union of its components unless \( P = [0, 1] \), and either (1) \( A_1 = \{ a, c \} \) and \( A_2 = \{ b \} \) where \( a < b < c \) or (2) \( A_2 = \{ a, c \} \) and \( A_1 = \{ b \} \) where \( a < b < c \); it follows from the assumption that \( A_1, A_2, \) and \( P \) have the uniqueness property. Assume that sets \( p^{m} A_{1}^{k_1} + (1 - p^{m}) A_{2}^{k_2} \) are components of \( P_A + (1 - P)A_2 \).

Thus,
\[
G_{u,1/2}(P_A + (1 - P)A_2)
\]
is the union of its components unless it is the union of its components unless \( P = [0, 1] \), and either (1) \( A_1 = \{ a, c \} \) and \( A_2 = \{ b \} \) where \( a < b < c \) or (2) \( A_2 = \{ a, c \} \) and \( A_1 = \{ b \} \) where \( a < b < c \); it follows from the assumption that \( A_1, A_2, \) and \( P \) have the uniqueness property. Assume that sets \( p^{m} A_{1}^{k_1} + (1 - p^{m}) A_{2}^{k_2} \) are components of \( P_A + (1 - P)A_2 \).

Thus,
\[
G_{u,1/2}(P_A + (1 - P)A_2)
\]
is the representation of \( P_A + (1 - P)A_2 \) as the union of its components unless \( P = [0, 1] \), and either (1) \( A_1 = \{ a, c \} \) and \( A_2 = \{ b \} \) where \( a < b < c \) or (2) \( A_2 = \{ a, c \} \) and \( A_1 = \{ b \} \) where \( a < b < c \); it follows from the assumption that \( A_1, A_2, \) and \( P \) have the uniqueness property. Assume that sets \( p^{m} A_{1}^{k_1} + (1 - p^{m}) A_{2}^{k_2} \) are components of \( P_A + (1 - P)A_2 \).

Thus,
\[
G_{u,1/2}(P_A + (1 - P)A_2)
\]
is the representation of \( P_A + (1 - P)A_2 \) as the union of its components unless \( P = [0, 1] \), and either (1) \( A_1 = \{ a, c \} \) and \( A_2 = \{ b \} \) where \( a < b < c \) or (2) \( A_2 = \{ a, c \} \) and \( A_1 = \{ b \} \) where \( a < b < c \); it follows from the assumption that \( A_1, A_2, \) and \( P \) have the uniqueness property. Assume that sets \( p^{m} A_{1}^{k_1} + (1 - p^{m}) A_{2}^{k_2} \) are components of \( P_A + (1 - P)A_2 \).

Thus,
\[
G_{u,1/2}(P_A + (1 - P)A_2)
\]
is the representation of \( P_A + (1 - P)A_2 \) as the union of its components unless \( P = [0, 1] \), and either (1) \( A_1 = \{ a, c \} \) and \( A_2 = \{ b \} \) where \( a < b < c \) or (2) \( A_2 = \{ a, c \} \) and \( A_1 = \{ b \} \) where \( a < b < c \); it follows from the assumption that \( A_1, A_2, \) and \( P \) have the uniqueness property. Assume that sets \( p^{m} A_{1}^{k_1} + (1 - p^{m}) A_{2}^{k_2} \) are components of \( P_A + (1 - P)A_2 \).

Thus,
\[
G_{u,1/2}(P_A + (1 - P)A_2)
\]
is the representation of \( P_A + (1 - P)A_2 \) as the union of its components unless \( P = [0, 1] \), and either (1) \( A_1 = \{ a, c \} \) and \( A_2 = \{ b \} \) where \( a < b < c \) or (2) \( A_2 = \{ a, c \} \) and \( A_1 = \{ b \} \) where \( a < b < c \); it follows from the assumption that \( A_1, A_2, \) and \( P \) have the uniqueness property. Assume that sets \( p^{m} A_{1}^{k_1} + (1 - p^{m}) A_{2}^{k_2} \) are components of \( P_A + (1 - P)A_2 \).

Thus,
\[
G_{u,1/2}(P_A + (1 - P)A_2)
\]
is the representation of \( P_A + (1 - P)A_2 \) as the union of its components unless \( P = [0, 1] \), and either (1) \( A_1 = \{ a, c \} \) and \( A_2 = \{ b \} \) where \( a < b < c \) or (2) \( A_2 = \{ a, c \} \) and \( A_1 = \{ b \} \) where \( a < b < c \); it follows from the assumption that \( A_1, A_2, \) and \( P \) have the uniqueness property. Assume that sets \( p^{m} A_{1}^{k_1} + (1 - p^{m}) A_{2}^{k_2} \) are components of \( P_A + (1 - P)A_2 \).
Thus,
\[
\left( |S| + |T_1| + |T_2| \right) G_{u,1/2}(A_1 \cup A_2)
\]
\[
= \sum_{s \in S} \frac{b_s + a_s}{2} + \sum_{t \in T_1} \frac{f_{1t} + e_{1t}}{2} + \sum_{t \in T_2} \frac{f_{2t} + e_{2t}}{2}
\]
\[
= \sum_{s \in S} \left\{ \frac{n(s,3-i) p_s^{3-i}}{2} + \sum_{k=1}^{2-i} \frac{d_{sk}^1 + e_{sk}^1}{2} - \sum_{k=1}^{2-i} \frac{d_{sk}^2 + e_{sk}^2}{2} \right\}
\]
\[
+ \sum_{t \in T_1} \frac{f_{1t} + e_{1t}}{2} + \sum_{t \in T_2} \frac{f_{2t} + e_{2t}}{2} + \sum_{s \in S} \left\{ \frac{k(s,1) c_{sk}^1 + e_{sk}^1}{2} + \sum_{k=1}^{2} \frac{d_{sk}^2 + e_{sk}^2}{2} \right\}
\]
\[
- \sum_{s \in S} \left\{ \frac{k(s,1) a_{sk}^1 + e_{sk}^1}{2} + \sum_{k=1}^{2} \frac{k(s,2) d_{sk}^2 + e_{sk}^2}{2} \right\} = \left[ \sum_{s \in S} n(s,1) + |T_1| + \sum_{s \in S} k(s,1) \right] G_{u,1/2}(A_1)
\]
\[
+ \left[ \sum_{s \in S} n(s,2) + |T_2| + \sum_{s \in S} k(s,2) \right] G_{u,1/2}(A_2)
\]
\[
- \left[ \sum_{s \in S} (n(s,1) + n(s,2) - 1) + \sum_{s \in S} (k(s,1) + k(s,2)) \right] G_{u,1/2}(A_1 \cap A_2)
\]
\[
\leq \left( |S| + |T_1| + |T_2| \right) G_{u,1/2}(A_2),
\]
where the inequality follows from the assumption that \( G_{u,1/2}(A_1) \leq G_{u,1/2}(A_2) \) and \( G_{u,1/2}(A_2) \leq G_{u,1/2}(A_1 \cap A_2) \).

Acknowledgements. I am grateful to Walter Bossert and Peter Klibanoff for their remarks and suggestions regarding related literature. I am grateful to the workshop and conference participants, especially Eddie Dekel and Marciano Siniscalchi, for their comments. I would also like to thank the editor and two referees for valuable suggestions. I thank the National Science Foundation (grant SES-0453061) for financial support.

REFERENCES


DEKEL, E., LIPMAN, B. L. and RUSTICHINI, A. (2004), “Temptation-Driven Preferences” (Mimeo, Northwestern and Tel-Aviv University, Boston University and University of Minnesota).


